The Hodge decomposition theorem

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0 Introduction

The aim of this note is to prove the Hodge decomposition theorem for compact Kähler manifolds. On any complex manifold, the complex valued $k-$forms split as $\Omega_X^k = \bigoplus \Omega^{p,q}$. The Hodge decomposition tells us that this holds when we pass to cohomology as well, but only for a special class of complex manifolds (importantly, there are complex manifolds for which the decomposition does not hold, e.g. Hopf surfaces)

The outline will be the following: we start first with a brief review of the local theory, i.e. complex analysis of several variables.

Then we move to the definition of a complex manifold and prove various things about the complexified tangent bundle and the holomorphic tangent bundle. We mention almost complex structures and the Newlander-Nirenber theorem only briefly, as we will almost exclusively be dealing with compact Kähler manifolds. We define the ∂ and $\overline{\partial}$ operators, the latter of which is used to define Dolbeault cohomology. We also prove a Poincare-type lemma for the Dolbeault complex, which will be essential in showing the Dolbeault complex gives us a resolution of sheaves.

Then we will define Kähler manifolds and show in particular that complex projective space (and all its submanifolds, i.e. projective manifolds) carries a Kähler metric, the Fubini-Study metric.

After that, we deal with sheaves and sheaf cohomology, and their relation to vector bundles. A lot of the proofs have been omitted, but they can be found in most books on algebraic geometry or homological algebra.

Finally, the last section is devoted to proving the Hodge decomposition theorem. For this, we need to define the Hodge star operator, Laplacians and harmonic forms, as well as the Kähler identities. The Hodge decomposition theorem then follows using an application of a difficult theorem on elliptic partial differential operators.

1 Review of complex analysis, local theory

We begin by defining the basic objects of complex geometry, namely differential forms and holomorphic functions in the case of $\mathbb{C}^n$. This will be done by taking the usual cotangent bundle, complexifying it and doing a lot of linear algebra.

1.1 Complex differential forms and holomorphic functions

Given $U \subseteq \mathbb{C}$ open, we can think of U as an open subset of $\mathbb{R}^2$ and we have the usual C^∞ tangent bundle T_U, whose sections are the smooth vector fields. Now define $dz = dx + idy$, which takes in values in T_U and lands in $\mathbb{C}$. In other words, it is a differential 1-form with coefficients in $\mathbb{C}$, $dz \in T_U^* \otimes \mathbb{C}$. Moreover, its conjugate is $d\bar{z} = dx - idy$. Hence dz and $d\bar{z}$ form a basis of this complexified cotangent bundle which we denote by $\Omega_{U,\mathbb{C}}$. We have that

$$dx = \frac{1}{2}(dz + d\bar{z}),\ dy = \frac{-i}{2}(dz - d\bar{z})$$

Taking duals, we get that

$$\frac{\partial}{\partial x} = \frac{1}{2}(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}), \frac{\partial}{\partial y} = \frac{-i}{2}(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}})$$

We can extend the usual exterior derivative $\mathbb{C}$-linearly, and for a smooth function f, its derivative will end up in the complexified cotangent bundle, whose basis consists of dz and $d\bar{z}$. If we put $df = f_z dz + f_{\bar{z}} d\bar{z}$, then evaluating at $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively tells us that $\frac{\partial f}{\partial x} = f_z + f_{\bar{z}}$ and $\frac{\partial f}{\partial y} = if_z - if_{\bar{z}}$. By elimination, we compute that

$$\begin{aligned} \frac{\partial f}{\partial z} &:= f_z = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) \\ \frac{\partial f}{\partial \bar{z}} &:= f_{\bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) \end{aligned} \tag{1.1}$$

Observation: *We can interpret the Cauchy-Riemann equations as precisely equivalent to the equation $\frac{\partial f}{\partial \bar{z}} = 0$. In other words, if f is holomorphic, then $df = f_z dz$.*

Now let us see what happens to the matrix of the complexified exterior derivative. The usual Jacobian matrix for the differential of f is given by $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$. Now, i acts on $\mathbb{R}^2$ as the matrix $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (we will define this later as an almost complex structure). The Cauchy-Riemann equations can now be seen to be equivalent to the commuting of the Jacobian df with the endomorphism j. In other words, a function f is holomorphic if and only if its differential is not only $\mathbb{R}$-linear, but $\mathbb{C}$-linear!

Extending this to the multivariable case, the real and complex Jacobians of a smooth function f

are related as following: the usual real Jacobian of f with respect to the basis $\{\frac{\partial}{\partial x_i}.\frac{\partial}{\partial y_i}\}$ is

$$J_{\mathbb{R}}(f) = \begin{pmatrix} (\frac{\partial u_i}{\partial x_j})_{i,j} & (\frac{\partial u_i}{\partial y_j})_{i,j} \\ (\frac{\partial v_i}{\partial x_j})_{i,j} & (\frac{\partial v_i}{\partial y_j})_{i,j} \end{pmatrix}$$

This can be extended to a $\mathbb{C}$-linear map with the same determinant, namely (this time with respect to the basis $\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\}$):

$$\begin{pmatrix} (\frac{\partial f_i}{\partial z_j})_{i,j} & (\frac{\partial f_i}{\partial \bar{z}_j})_{i,j} \\ (\frac{\partial \bar{f}_i}{\partial z_j})_{i,j} & (\frac{\partial \bar{f}_i}{\partial \bar{z}_j})_{i,j} \end{pmatrix} \tag{1.2}$$

Definition 1.1: *A smooth function $f : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is holomorphic if and only if its differential commutes with the almost complex structures on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$. In other words, $df_u : T_u\mathbb{R}^{2n} \to T_{f(u)}\mathbb{R}^{2m}$ is $\mathbb{C}$-linear, where we identify $T_u\mathbb{R}^{2n} \simeq \mathbb{R}^{2n} \simeq \mathbb{C}^n$.*

In other words, when f is holomorphic, the matrix 1.2 reduces to

$$\begin{pmatrix} J_{\mathbb{C}}(f) & 0 \\ 0 & \bar{J}_{\mathbb{C}}(f) \end{pmatrix}$$

with $J_{\mathbb{C}}(f) = (\frac{\partial f_i}{\partial z_j})_{i,j}$ being the complex Jacobian. In this case we conclude that $det(J_{\mathbb{C}}(f))^2 = det(J_{\mathbb{R}}(f))$

Remark: There is an endomorphism on $\mathbb{R}^{2n}$ which sends $(x_i, y_i) \to (-y_i, x_i)$, known as an almost complex structure.

We now list some important properties of holomorphic functions.

Theorem 1.2 (Properties of holomorphic functions):

- *Compositions of holomorphic functions are holomorphic*
- *If f is holomorphic, then fdz is closed (resp. in the multivariable case)*
- *(Cauchy's integral formula) If f is holomorphic on U and D is a disk contained in U, then $f(z_0) = \frac{1}{2i\pi}\int_{\partial D}\frac{f(z)}{z-z_0}dz$ for any $z_0 \in D^o$, in the one-variable case. In the multivariable case, we have $f(z_1, z_2, ..., z_n) = \frac{1}{2i\pi}^n\int_{\Lambda} f(\zeta)\frac{d\zeta_1)}{\zeta_1 - z_1}...\frac{d\zeta_n)}{\zeta_n - z_n}dz$ for any $z_0 \in D^o$, where Λ is a product of circles.*
- *As a corollary to the Cauchy Integral formula, we have that every holomorphic function is locally analytic, i.e. expressible as a power series.*
- *Maximum principle: If f is holomorphic on an open U and $|f|$ admits a local maximum on U, then f is locally constant around it.*
- *Zeros are isolated: If f vanishes on an open subset of U, then it is identically zero.*

One way to prove the Cauchy integral formula uses Stokes' theorem.

Theorem 1.3 (Stokes' theorem): *Given a smooth* $n-1$ *form* ω *on an* n*-dimensional smooth manifold with boundary* M*, we have that* $\int_M d\omega = \int_{\partial M} \omega$

Another useful fact is that the operator $\frac{\partial}{\partial \bar{z}}$ is *locally exact.* This is a sort of holomorphic Poincare lemma and is important, as it makes the maps in the Dolbeault resolution exact.

Theorem 1.4 (Local exactness of $\bar{\partial}$): *If* f *is some* C^k *function on an open set* U *of* $\mathbb{C}$*, then there exists some* C^k *function* g *such that, locally,* $\frac{\partial g}{\partial \bar{z}} = f$ *.*

Proof. This can be shown by considering $g(z) = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$ □

2 Complex Manifolds

Now we move on to defining complex manifolds. These are spaces which have holomorphic atlases:

Definition 2.1 (Complex manifold): *A complex manifold X is a smooth manifold of even dimension such that the transition maps are not only smooth, but holomorphic as functions on open subsets of $\mathbb{C}^n$.*

Using this definition, the holomorphic functions on a complex manifold are defined as functions which become holomorphic when precomposed with the trivializations, i.e. if $\phi : U \simeq U' \subseteq \mathbb{C}^n$ is a local trivialization and $f : X \to \mathbb{C}$, then we require that $f|_U \circ \phi^{-1} : U' \to \mathbb{C}$ is holomorphic.

Remark: By the maximum principle, any holomorphic function on a compact connected complex manifold is constant. This also implies that the equivalent of the Whitney embedding theorem fails for complex manifolds, as compact complex manifolds cannot be embedded in $\mathbb{C}^n$, since the projection maps are holomorphic and nonconstant.

The data of the open cover and holomorphic transition maps is called a **complex structure**. This should not be confused with an **almost complex structure**, which is an endomorphism of the tangent bundle whose square is -1, to be defined in the next section. However, every complex manifold carries with it an associated almost complex structure, and the converse question has an answer in the Newlander-Nirenber theorem.

2.1 The almost complex structure on a complex manifold

To show the existence of an almost complex structure on X, we first define a canonical almost complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$, give it to the patches of X and then glue them together.

Take the canonical almost complex structure on $\mathbb{C}^n = \mathbb{R}^{2n}$ sending

$$(x_1, y_1, ..., x_n, y_n) \mapsto (-y_1, x_1, ..., -y_n, x_n)$$

Now define $I : T\mathbb{C}^n \to T\mathbb{C}^n$ to be the induced almost complex structure on the (smooth) tangent bundle of $\mathbb{C}^n$ by using the identifications $T_x\mathbb{C}^n \simeq \mathbb{C}^n$.

If $\phi : U \to \mathbb{C}^n, \psi : V \to \mathbb{C}^n$ are local trivializations of some U, V in a cover forming a complex structure on X, then the transition maps τ_{UV} are holomorphic. Hence $d\phi$ identifies T_U with $U \times \mathbb{C}^n$ and similarly with V, and does so in such a way that, at a given point x, the holomorphic map $d\tau_{UV}$ is $\mathbb{C}-$linear.

Now we can give almost complex structures $J^U \in End(T_U)$ and $J^V \in End(T_V)$ along the maps $d\phi$ and $d\psi$. More precisely, $J^U = d\phi^{-1} \circ I \circ d\phi, J^V = d\psi^{-1} \circ I \circ d\psi$. These glue together along $U \cap V$ for the following reason: on $U \cap V$, $d\phi = d\tau_{UV} \circ d\psi$ and hence $J^U = d\psi^{-1} \circ d\tau_{UV}{}^{-1} \circ I \circ d\tau_{UV} \circ d\psi = d\psi^{-1} \circ I \circ d\psi = J^V$ since by definition 1.1, the holomorphicity of τ_{UV} means that $d\tau_{UV}$ commutes with I.

All in all, the separate I^U's glue together to form $J : T_X \to T_X$ with $J^2 = -1$, which is the almost complex structure on the tangent bundle of X.

2.2 The holomorphic tangent bundle

We will define the holomorphic tangent bundle associated to a complex manifold X.

Firstly, let's remind ourselves of the smooth tangent bundle $T_{X,\mathbb{R}}$ associated to a smooth manifold X (I omitted the subscript $\mathbb{R}$ in the previous section to simplify notation). This can be defined in a variety of ways: jets, derivations on the algebra of smooth functions, or most conveniently for our purposes, using a cocycle.

One way to do this is to define $T_X = \coprod_{p\in X} T_pX$ where T_pX consists of the derivations at the point $p \in X$. This is a vector bundle over X by projecting a derivation at p to p since each T_pX is isomorphic to $\mathbb{R}^k$. Consider a smooth structure for X consisting of an open cover $\{U_i\}$. Then $\pi^{-1}(U_i) = \coprod_{p\in U_i} T_pX \simeq U_i \times \mathbb{R}^k$ via $d\phi_{U_i}$, where $\phi_{U_i} : U_i \simeq \mathbb{R}^k$, i.e. $T_pX = T_pU_i \simeq T_{\phi_{U_i}p}\mathbb{R}^k \simeq \mathbb{R}^k$. Looking more carefully, the cocycle associated to this vector bundle consists of the differentials of the transition maps which are exactly the Jacobian matrices $\psi_{ij} = d(\phi_j\phi_i^{-1})$, which means that we can construct

$$T_{X,\mathbb{R}} = \coprod U_i \times \mathbb{R}^k / \sim,$$

where $(x, v) \sim (x, J_{\mathbb{R}}(\tau_{ij})_x(v))$ for $x \in U_i \cap U_j$. In other words, we get a vector bundle that is trivial over the open sets U_i. However, we can do the exact same thing, replacing $\mathbb{R}$ with $\mathbb{C}$, smooth functions with holomorphic functions and the real Jacobian with the complex one $\frac{\partial f^k}{\partial z^j}$ to produce the holomorphic tangent bundle.

Definition 2.2 (Holomorphic tangent bundle): *The holomorphic tangent bundle T_X associated to a complex manifold X is the complex vector bundle associated to the cocycle given by the complex Jacobians of the transition maps.*

$$T_X = \coprod U_i \times \mathbb{C}^n / \sim$$

It is also useful to have a more direct identification between $T_w\mathbb{C}^n$ and $\mathbb{C}^n$. This can be done by defining $T_w\mathbb{C}^n$ as the complex-valued algebra of point-derivations on the complex algebra of holomorphic functions, which can be shown to be generated by the $\frac{\partial}{\partial z_i}$ as a complex vector space.

Here's a sketch of how this goes in the one variable case: if $w \in \mathbb{C}$ is a point and D is a point-derivation at w, then $Df = \tilde{D}u + i\tilde{D}v$ where by abuse of notation we put f as the germ of $f = u+iv$ at w and $\tilde{D}$ is the induced real point-derivation on the functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. But the real point derivations are generated by $\frac{\partial}{\partial x}|_w$, $\frac{\partial}{\partial y}|_w$ and so we have $\tilde{D} = \alpha\frac{\partial}{\partial x}|_w + \beta\frac{\partial}{\partial y}|_w$ for some $\alpha, \beta \in \mathbb{R}$. But the holomorphicity of f means, using the Cauchy-Riemann equations, that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Plugging this in, we get that

$$Df = (\alpha + i\beta)(\frac{\partial u}{\partial x}|_w - i\frac{\partial u}{\partial y}|_w),$$

whereas $\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x}|_w - i\frac{\partial u}{\partial y}|_w$, so we see that $D = (\alpha + i\beta)\frac{\partial}{\partial z}$. In other words, the derivations at the point w are generated by $\frac{\partial}{\partial z}$ as a complex vector space.

2.3 Comparing the holomorphic and smooth tangent bundles

The relationship between the differentiable and holomorphic vector bundles is the following: let $T_{X,\mathbb{R}}$ be the differentiable vector bundle. This acquires a structure as a complex vector space induced by the almost complex structure J. Then we can complexify it: $T_{X,\mathbb{R}} \otimes \mathbb{C}$ and we further have an inclusion $T_X \subset T_{X,\mathbb{R}} \otimes \mathbb{C}$, by equation 1.1. (Beware: the complexified differentiable tangent bundle has two almost complex structures, one given by $J \otimes Id$ and one given by i. We are using the first one). The operator J has two eigenspaces corresponding to $\pm i$ and if we denote them by $T_X^{1,0}$ and $T_X^{0,1}$, we get the splitting

$$T_{X,\mathbb{R}} \otimes \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$$

By linear algebra, $T_X^{1,0} = \{u - iJu | u \in T_{X,\mathbb{R}}\}, T_X^{0,1} = \{u + iJu | u \in T_{X,\mathbb{R}}\}$. Locally, i.e. when $X = \mathbb{C}^n$, the real tangent bundle is generated by $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$. Hence, $T_X^{1,0}$ is generated by

$$\frac{\partial}{\partial x_j} - iJ\frac{\partial}{\partial x_j} = 2\frac{\partial}{\partial z_j}, \frac{\partial}{\partial y_j} - iJ\frac{\partial}{\partial y_j} = 2i\frac{\partial}{\partial z_j}$$

Similarly, $T_X^{0,1}$ is generated by $\frac{\partial}{\partial \overline{z_j}}$, i.e. $T_X^{1,0} = \overline{T_X^{0,1}}$. Since the holomorphic tangent bundle is also locally generated by $\frac{\partial}{\partial z_j}$, we can identify the holomorphic tangent bundle T_X and $T_X^{1,0}$ as complex vector bundles.

Dualizing everything, we get the decomposition

$$\Omega_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

of the complexified differential forms (**not** the holomorphic ones!) and more generally

$$\Omega_{X,\mathbb{C}}^k = \bigwedge^k \Omega_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q}$$

with $\Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \oplus \bigwedge^q \Omega_X^{0,1}$. Note that this is a complex vector bundle, and not a holomorphic one.

In local coordinates, $\Omega_X^{1,0}$ is generated by the dz_j, whereas $\Omega_X^{0,1}$ is generated by the $d\bar{z}_j$. Hence, a basis for $\Omega_X^{p,q}$ is given by $\{dz_{i_1} \wedge ... \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q} | \, i_1 < ... < i_p, j_1 < ... < j_q\}$.

2.4 The operators ∂ and $\overline{\partial}$

The complexified exterior differential on complex-valued differential forms sends a form of type (p,q) to a sum of forms of types $(p,q+1)$ and $(p+1,q)$. The first one we denote by $\overline{\partial}$ and the second by ∂. Hence we have $d = \partial + \overline{\partial}$. On 0-forms, this looks as follows: let f be some differentiable function $X \to \mathbb{C}$. Then $df = \sum \frac{\partial f}{\partial z_j} dz_j + \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j = \partial f + \overline{\partial} f$. More generally, one can extend this, using the Leibniz rule, to all forms, by the formula

$$d\alpha = \sum d\alpha_{I,J} dz_I \wedge d\bar{z}_J = \sum \frac{\partial \alpha_{I,J}}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J + \sum \frac{\partial \alpha_{I,J}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J$$

Proposition 2.3:

- ∂ *and* ∂ *obey the Leibniz rule*
- $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$

Proof. This follows from the corresponding property of d and looking at the bidegree. □

The operator $\overline{\partial}$ is a map $C^\infty(U, \Omega_X^{p,q}) \to C^\infty(U, \Omega_X^{p,q+1})$ (and similarly for ∂). By the previous proppposition, for any p these form a chain complex called the **Dolbeault** complex

$$0 \to C^\infty(U, \Omega_X^{p,0}) \to C^\infty(U, \Omega_X^{p,1}) \to ...$$

With $U = X$, the cohomology groups of this complex are known as the Dolbeault cohomology groups, denoted $H^q(C^\infty(X, \Omega_X^{p,-})) = H^{p,q}(X)$.

One can ask: since we have a decomposition

$$C^\infty(X, \Omega_{X,\mathbb{C}}^k) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p,q}),$$

does the same hold when we pass to cohomology, i.e. $H^k(X, \mathbb{C}) = \bigoplus H^{p,q}(X)$? This does not hold always, and is the main topic of this note. The **Hodge decomposition theorem**, which we will prove, says that it holds for the class of all Kähler manifolds.

Now we show a Poincare lemma for the operator $\overline{\partial}$.

Theorem 2.4 (Exactness of Dolbeault resolution): *Any* $\overline{\partial}$ *closed form* $\alpha \in C^\infty(U, \Omega_X^{p,q})$ *is locally exact.*

Proof. First reduce to the case where α is of type $(0, q)$ as follows: write $\alpha = \sum \alpha_{I,J} dz_I \wedge d\bar{z}_J$. Hence

$$0 = \overline{\partial}\alpha = \sum \frac{\partial \alpha_{I,J}}{\partial \bar{z}_j} z_j \wedge dz_I \wedge d\bar{z}_J$$

Now this means that each $\alpha_I = \sum \alpha_{I,J} d\bar{z}_J$ is $\overline{\partial}$-closed. If we have proved the case for forms of type $(0, q)$, then locally $\alpha_I = \overline{\partial}\beta_I$ and then $\alpha = \pm\overline{\partial}(\sum dz_I \wedge \beta_I)$. Hence we only need to prove the case where $p = 0$, which follows by induction from theorem 1.4 □

Let's generalize the previous section to an arbitrary rank k complex vector bundle E over X. Let $A^{0,q}(E) = C^\infty(X, E \otimes \Omega_X^{0,q})$, the differential k-forms with coefficients in E. Locally on a trivializing open set U, an element α in $A^{0,q}(E)$ i.e. a section of $E \otimes \Omega_X^{0,q}$ will look like a k-tuple of $(0,q)$-forms i.e. $\alpha = (\alpha_1, ..., \alpha_k)$. If we define $\overline{\partial}_U \alpha$ by $(\overline{\partial}\alpha_1, ..., \overline{\partial}\alpha_k)$, then all of these $\overline{\partial}_U$ glue together, since the transition matrices have holomorphic coefficients[1], forming an exterior derivative

$$\overline{\partial}_E : A^{0,q}(E) \to A^{0,q+1}(E)$$

Definition 2.5 (Dolbeault complex of a vector bundle): *The Dolbeault complex of a vector bundle is the complex*

$$0 \to A^{0,0}(E) \xrightarrow{\overline{\partial}_E} A^{0,1}(E) \to ...$$

[1]For more details, consult Voisin, Lemma 2.33

3 Kähler manifolds, differential operators

In this section we introduce a large class of complex manifolds for which we will prove the Hodge decomposition, namely the Kähler manifolds.

3.1 Hermitian structures and Kähler forms

Let V be a complex vector space and $V_{\mathbb{R}}$ be the underlying real vector space and $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ its complexification. Further, let $W = \text{Hom}(V, \mathbb{R})$ be the dual of V and define $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$ similarly. These vector spaces have an almost complex structure and split as before:

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}, W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$$

The second exterior power splits as

$$\bigwedge^2 W_{\mathbb{C}} = \bigwedge^2 W^{1,0} \oplus (W^{1,0} \wedge W^{0,1}) \oplus \bigwedge^2 W^{0,1}$$

We denote the middle part as $W^{1,1}$ and by $W^{1,1}_{\mathbb{R}}$ its real part.

A Hermitian form on V is a map $V \times V \to \mathbb{C}$ that is $\mathbb{C}$-linear in the first variable, $\mathbb{C}$-antilinear in the second and satisfies $h(u, v) = \overline{h(v, u)}$. There is a correspondence between real (1,1)-forms and Hermitian forms, as the following proposition shows.

Proposition 3.1 (Hermitian forms and 1-1 forms): *There is a bijection between Hermitian forms on V and real (1,1) forms on V given by $h \mapsto -Im(h)$ and $\omega \mapsto h$ with $h(u, v) = \omega(u, Jv) - i\omega(u, v)$*

Proof. Since h is Hermitian, we see that $\omega = -Im(h)$ is alternating. It is of type $(1, 1)$ since $\omega(u - iJu, v - iJv) = 0$ (since h is Hermitian) i.e. it vanishes on pairs of elements of $V^{1,0}$ and similarly with $V^{0,1}$.

To show the converse, put $h(u, v) = \omega(u, Jv) - i\omega(u, v)$. Then $h(u, Jv) = \omega(u, -v) - i\omega(u, Jv) = -\omega(u, v) - i\omega(u, Jv) = -ih(u, v)$. Since ω is of type $(1, 1)$, we have that $\omega(u, Jv) = -\omega(Ju, v)$ using the fact that $0 = \omega(u - iJu, v - iJv) = [\omega(u, v) - \omega(Ju, Jv)] - i[\omega(u, Jv) + \omega(Ju, v)]$ and looking at the imaginary part. Now, using the fact that ω is alternating, we have

$$h(v, u) = \omega(v, Ju) - i\omega(v, u) = \omega(u, Jv) + i\omega(u, v) = \overline{h(u, v)}$$

In other words, h is Hermitian, as desired. □

In coordinates, fixing a basis $z_1, .., z_n$ of V, denote $h(z_i, z_j) = h_{ij}$. Then if $u = (u_1, ..., u_n), v = (v_1, ..., v_n)$, we have $h(u, v) = \sum h_{ij} u_i \overline{v}_j$ and can thus write

$$h = \sum h_{ij} z_i^* \otimes \overline{z_j}^*$$

Now,

$$\omega(u,v) = -Im(h(u,v)) = \frac{-1}{2i}(h(u,v) - \overline{h(u,v)}) = \frac{i}{2}(h(u,v) - h(v,u))$$

i.e.

$$\omega(u,v) = \frac{i}{2}\sum h_{ij}(u_i\overline{v}_j - v_i\overline{u}_j)$$

which means we can identify

$$w = \frac{i}{2}\sum h_{ij} z_i^* \wedge \overline{z}_j^*$$

since $[z_i^* \wedge \overline{z}_j^*](u,v) = u_i\overline{v}_j - v_i\overline{u}_j$.

Definition 3.2 (Hermitian metric and Kähler forms): *A Hermitian metric on a complex manifold X is a collection h_x of Hermitian forms on the holomorphic tangent space T_xX. To such a collection of metrics we can associate the real 2-form of type $\omega = -Im(h) \in \Omega^2_{X,\mathbb{R}} \cap \Omega^{1,1}_X$, called the fundamental form. This form is called Kähler if it is closed.*

The canonical example of a Kähler form is the 2-form associated to the standard inner product on $\mathbb{C}^n$, namely the form $\omega = \frac{i}{2}\sum dz_i \wedge d\overline{z}_i$. In fact, the fundamental form of a Kähler manifold locally behaves approximately like the associated form of the standard Hermitian metric[2].

Moreover, the fundamental form is related to the volume form of a complex manifold. Given a complex manifold X with a Hermitian metric h, let $e_1, ..., e_n \in T_xX$ be an orthonormal basis for h_x over $\mathbb{C}$. Then $e_1, Je_1, ..., e_n, Je_n$ is orthonormal for $g_x = Re(h_x)$, the Riemannian metric. Denote the dual basis for $\Omega_{X,x,\mathbb{R}}$ to be $dx_1, dy_1, ..., dx_n, dy_n$ and put $dz_i = dx_i + idy_i$. We have

$$w_x = \frac{i}{2}\sum dz_i \wedge d\overline{z}_i$$

and hence

$$\frac{w_x^n}{n!} = (\frac{i}{2})^n dz_1 \wedge d\overline{z}_1 ... \wedge dz_n \wedge d\overline{z}_n = dx_1 \wedge dy_1 ... \wedge dx_n \wedge dy_n$$

and so $w^n/n!$ is a volume form. In particular,

$$\text{Vol}(X) = \int_X \frac{w^n}{n!} > 0$$

If ω is Kähler, then ω^k is closed for all k, so it defines a nontrivial De Rham cohomology class in $H^{2k}(X,\mathbb{R})$ for the following reason: if $\omega^k = d\eta$, then $\omega^n = d(\omega^{n-k} \wedge \eta)$ and by Stokes' theorem we will get that $\text{Vol}(X) = 0$, which is impossible. Hence, if a manifold is Kähler, it has to have nontrivial de Rham cohomology groups in all even dimensions.

3.2 Projective space and the Fubini-Study metric

In this section, we show that any projective manifold is Kähler, using the Fubini-Study metric[3].

[2] See Proposition 3.14 in Voisin

[3] Note that the Fubini-Study metric can also be defined as the Chern form associated to the tautological line bundle over projective space - see Voisin 3.3.1

Recall that complex projective space $\mathbb{P}^n$ has an open cover consisting of $U_i = \{(w_0 : ... : w_n)|z_i \neq 0\} \simeq \mathbb{C}^n$ sending $(w_0 : ... : w_n) \mapsto (\frac{w_0}{w_i}, ..., \frac{w_n}{w_i})$ (omitting the i-th component which is just 1). On this patch, define

$$\omega_i = \frac{i}{2\pi}\partial\overline{\partial}\log(\sum|\frac{w_j}{w_i}|^2)$$

Under the trivialization $\phi_i : U_i \to \mathbb{C}^n$, this corresponds to

$$\frac{i}{2\pi}\partial\overline{\partial}\log(1 + \sum|z_k|^2)$$

We want to show that these glue to a global closed form on $\mathbb{P}^n$. But

$$\log(\sum|\frac{w_j}{w_i}|^2) = \log(|\frac{w_k}{w_i}|^2\sum|\frac{w_j}{w_k}|^2)) = \log(|\frac{w_k}{w_i}|^2) + \log(\sum|\frac{w_j}{w_k}|^2)$$

So to show ω_i and ω_k agree on $U_k \cap U_i$, we need to show that

$$\partial\overline{\partial}\log(|\frac{w_k}{w_i}|^2) = 0$$

When $i < k$, on U_i the function w_k/w_i corresponds to the $k-$th coordinate on $\mathbb{C}^n$ under ϕ_i. But

$$\partial\overline{\partial}\log(|z|^2) = \partial(\frac{1}{z\bar{z}}\overline{\partial}(z\bar{z})) = \partial(\frac{zd\bar{z}}{z\bar{z}}) = \partial(\frac{d\bar{z}}{\bar{z}}) = 0$$

Hence the ω_i glue together to a global form ω. Moreover, it is clear that $d\omega = \partial\omega = \overline{\partial}\omega = 0$, using the fact that $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$, so ω is closed. Moreover, $\overline{w_i} = w_i$ using the fact that $\overline{\partial\overline{\partial}} = \overline{\partial}\partial = -\partial\overline{\partial}$, hence ω is real.

To show that ω arises from a metric, we need to show that its matrix is positive semidefinite. However,

$$\partial\overline{\partial}(1 + \sum|z_k|^2) = \frac{\sum dz_i d\bar{z}_i}{1 + \sum|z_i|^2} - \frac{(\sum \bar{z}_i dz_i) \wedge (\sum z_i d\bar{z}_i)}{(1 + \sum|z_i|^2)^2} = \frac{1}{(1 + \sum|z_i|^2)^2}\sum h_{ij} dz_i d\bar{z}_j$$

Now $h_{ij} = (1 + \sum|z_i|^2)\delta_{ij} - \bar{z}_i z_j$ which is positive by using the Cauchy-Schwarz inequality:

$$x^t(h_{ij})\bar{x} = x^t I\bar{x} + x^t z^t \bar{z}x - x^t \bar{z}z^t \bar{x} = (x, x) + (x, x)(z, z) - |(x, z)|^2 > 0$$

This completes the demonstration that projective space admits a Kähler metric.

Remark: The $\partial\overline{\partial}$ is not coincidental: after showing the Hodge decomposition for Kähler manifolds, one can prove the so-called $\partial\overline{\partial}$-lemma, which says that any d-closed form is locally $\partial\overline{\partial}$-exact.

Remark: Any complex submanifold of a Kähler manifold inherits the Hermitian metric and moreover inherits a Kähler form, and hence is Kähler. We have thus shown that any projective manifold is Kähler.

4 Sheaf cohomology

In this section we briefly recall some homological algebra and sheaf theory. We will explain how vector bundles correspond to locally free sheaves and also define the notion of sheaf cohomology.

4.1 Sheaves

The notion of a sheaf is a way to package all the local information of a geometric object. More precisely, a sheaf $\mathcal{F}$ assigns to each open subset U of a topological space X some object $\mathcal{F}(U)$(a set, group, ring, etc.). But it does not do so arbitrarily, but consistent with the structure of the topological space, i.e. whenever we have $U \subset V$, there is a restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$. A canonical example of a sheaf is the sheaf of sections of a vector bundle, and there the notion of restriction is very natural: if we have a section $s : V \to E$, then clearly restricting to U will give us another section, this time over U. It is for this reason that an element of $\mathcal{F}(X)$ is called a *global section.*

This data, with some extra conditions, defines what is called a **presheaf**. It can be described as a functor $\mathcal{O}(X)^{opp} \to \text{Set}$, where $\mathcal{O}(X)$ is the category of open subsets of X, with arrows being inclusion maps. A morphism of presheaves is a natural transformation between two such functors.

However, a sheaf is something more: it allows one to glue together sections defined on separate subsets, provided they agree on the intersections. This is called the gluing property:

Definition 4.1 (Sheaf gluing property): *Given sections $s_i \in \mathcal{F}(U_i)$ with U_i an open cover of U such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a unique global section $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$.*

In other words, a global section is given precisely by a coherent collection of local sections.

Definition 4.2 (Sheaf): *A sheaf abelian groups on a topological space X is a presheaf $\mathcal{F} : \mathcal{O}^{opp} \to \text{Ab}$ satisfying the sheaf gluing property and the identity property, i.e. the property that if we have two global sections $s, s' \in \mathcal{F}(X)$ that agree on an open cover of X i.e. $s|_U = s'|_U$, then $s = s'$.*

4.2 Vector bundles and locally free sheaves

Fundamental example: Over any complex manifold X, consider the trivial line bundle $X \times \mathbb{C}$. Then the sheaf of holomorphic sections of this line bundle consists of precisely the holomorphic functions on X. This sheaf is called the structure sheaf and is frequently denoted $\mathcal{O}_X$. Note that, locally, any holomorphic vector bundle E looks like $E_U \simeq X \times \mathbb{C}^n$ and so a holomorphic section $s \in \Gamma(U, E)$ can be multiplied pointwise by a holomorphic function: $f \cdot s(x) = f(x) \cdot s(x)$, where the multiplication comes from the structure of E_U as a complex vector space. Another way to say this is that the sheaf of sections of E is a sheaf of $\mathcal{O}_X$-modules! Moreover, the sheaf of sections of a holomorphic vector bundle locally looks like $\mathcal{O}_X^n$, where n is the rank of the vector bundle, i.e. it is locally free. This motivates the following:

Definition 4.3 (Locally free sheaves): *Let $\mathcal{A}$ be a sheaf of rings over X (i.e. the sheaf of continous, differentiable, holomorphic functions on a space etc.). Then a sheaf $\mathcal{F}$ is called a sheaf of $\mathcal{A}$-modules if each $\mathcal{F}(U)$ admits the structure of an $\mathcal{A}(U)$-module, compatible with the restriction maps. $\mathcal{F}$ is called a sheaf of free modules if there is a cover $\{U\}$ such that $\mathcal{F}(U) \simeq \mathcal{A}(U)^n$ for an integer n.*

We have shown that the sheaf of (continous, smooth, holomorphic etc.) sections of a vector bundle give examples of locally free sheaves. We now state a converse result, establishing a precise bijection between vector bundles and locally free sheaves.

Theorem 4.4 (Locally free sheaves and vector bundles): *Let $\mathcal{A}$ be a sheaf of functions and let E be a vector bundle. Then associating to E its sheaf of sections gives an equivalence between the locally free sheaves of $\mathcal{A}$-modules and vector bundles.*

Proof. We define the inverse map from locally free sheaves to vector bundles as follows: given a locally free sheaf $\mathcal{F}$, there is an open cover $\{U\}$ and a natural isomorphism $\tau_U : \mathcal{F}(U) \simeq \mathcal{A}(U)^n$. Hence, on an intersection $U \cap V$, we get an isomorphism $\tau_V \circ \tau_U^{-1} = \tau_{UV} : \mathcal{A}(U \cap V)^n \simeq \mathcal{A}(U \cap V)^n$. This isomorphism can be described by a matrix M_{UV} with entries in $\mathcal{A}(U \cap V)$ such that they satisfy the cocycle condition:

$$M_{UV} M_{VW} = M_{UW}$$

This allows us to construct a vector bundle that is trivial over the open sets U and with transition maps given by the matrices, i.e. $E = \coprod U \times \mathbb{F}^n / \sim$ (with $\mathbb{F}$ being the real or complex numbers) and $(x, v) \sim (x, w)$ iff $w = M_{UV} v$ and $x \in U \cap V$. The cocycle condition is in fact necessary for this to be a well-defined equivalence relation. □

4.3 Stalks and the sheaf associated to a presheaf

Now we move on to defining stalks, which track the local behaviour of sections on the microscopic level at a point p.

Definition 4.5 (Stalks): *The stalk of $\mathcal{F}$ at the point p is defined as the direct limit:*

$$\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U)$$

The direct limit can be constructed as a quotient of the coproduct $\coprod \mathcal{F}(U)$. What we are really doing is identifying two sections around p if they agree somewhere around p, in exact analogy with germs in differential geometry. In fact, knowing the microscopic behaviour of the sheaf at every point p can be used to reconstruct the whole sheaf as follows: one can construct the space of stalks $\coprod \mathcal{F}_p$ which is an etale space over X and one can recover $\mathcal{F}$ as the sheaf of sections of this etale space.

Note that the direct limit is functorial, so a morphism $\phi : \mathcal{F} \to \mathcal{G}$ induces a map $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$.

We have defined sheaves as special types of presheaves. However, every presheaf can be extended to a sheaf, which is called the sheafification of the presheaf (this is a free construction like any other in math and unsurprisingly forms a free-forgetful adjunction).

Definition 4.6 (Sheafification): *Given a presheaf $\mathcal{F}$ over X, the sheafification $\mathcal{F}^+$ is defined as follows:*

$$\mathcal{F}^+(U) = \{(f_p) \in \prod_{p \in U} \mathcal{F}_p | \textit{ for every } p \in U, \textit{there exists a neighbourhood } V \subset U$$
$$\textit{and a section } s \in \mathcal{F}(V) \textit{such that } s_q = f_q, \forall q \in V\}$$

The moral of the story is: collect all the stalks at points in U which locally arise as germs of sections in the presheaf. In this way, one generates new, glued sections which did not exist in the original presheaf, but which are needed for the presheaf to become a sheaf. There are other ways of defining the sheafification[4], but regardless of the construction, it is the universal sheaf in which a presheaf embeds. It is immediate also by the definition that the stalks of a presheaf and its associated sheaf are the same: using the same notation as the definition, the germ of any f on U can be restricted to some $V \subset U$ where it is represented by the section $s \in \mathcal{F}(V)$ and which in the stalk becomes just $s_p \in \mathcal{F}_p$. Hence $\mathcal{F}_p^+ = \mathcal{F}_p$.

Now we need to talk about how the category of sheaves over X forms what is called an **abelian category**, i.e. a category which has kernels, images and can talk about injectiveness, surjectiveness and exactness of short exact sequences. We begin with kernels.

Definition 4.7 (Injectivity and surjectivity of a morphism): *A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is called injective (resp. surjective) if it is injective (resp. surjective) at each point, $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$.*

Proposition 4.8 (Presheaf kernel is sheaf): *The kernel presheaf* $\ker(\phi)$ *sending $U \mapsto$* $\ker(\phi_U)$ *is actually a sheaf and is equal to* 0 *if and only if ϕ is injective.*

Proof. Clearly it is a presheaf. Furthermore, if $s_i \in \ker(\phi)_{U_i}$ are sections which are compatible,

[4] See Voisin Lemma 4.4, for example

then there is a global section $s \in \mathcal{F}_U$ which restricts to s_i. Then $\phi_U s \in \mathcal{G}_U$ restrics to 0 on an open cover of U, so must be 0.

For the second part, note that if ϕ is injective, then the stalk of the kernel sheaf at any point is the kernel of $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$, which is 0. Hence all the stalks are zero and the sheaf is also zero. □

The corresponding fact does not hold of images. However, modifying the situation by passing to the sheaf associated to the presheaf image, we have the follwoing:

Proposition 4.9 (Image sheaf): *The sheafification of the image presheaf* $\mathrm{im}(\phi)$ *is equal to* $\mathcal{G}$ *if and only if* ϕ *is surjective.*

Proof. Firstly, we have a sheaf map $j : \mathrm{im}(\phi)^+ \to \mathcal{G}$ by the universal property of the sheaf associated to a presheaf. But now, since the stalks of the sheafification are the same as the stalks of the original presheaf, j is injective, as it is already injective on the presheaf level. If ϕ is surjective, by the same reasoning j is surjective as well. Hence, for $\sigma \in \mathcal{G}_U$ there is an open cover og V and sections τ_V of $\mathrm{im}(\phi)^+$ such that $j_V \tau_V = \sigma_V$. Now

$$j_{U\cap V}(\tau_V)|_{U\cap V} = (j_V \tau_V)|_{U\cap V} = (j_U \tau_U)|_{U\cap V} = j_{U\cap V}(\tau_U)|_{U\cap V}$$

But j is injective, so we must have that $(\tau_V)|_{U\cap V} = (\tau_U)|_{U\cap V}$, which by the gluing property implies we have a global section τ which maps to σ and hence j is an isomorphism. The converse is immediate. □

Fundamental example: If we denote by $\mathcal{O}_X^*$ the sheaf of invertible holomorphic maps, where the groups are multiplicative, then the exponential map provides us with a locally surjective map

$$\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$$

i.e. a surjective morphism of sheaves with kernel the locally constant functions. However, when we pass to global sections, the exponential map is no longer surjective. The main idea behind sheaf cohomology is to fix this failure of exactness of the global sections functor by introducing the sheaf cohomology groups. In other words, on the level of sheaves we have the short exact exponential sequence (we will define this precisely in a moment)

$$0 \to 2i\pi\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

which will give us a long exact sequence

$$0 \to H^0(2i\pi\mathbb{Z}) \to H^0(\mathcal{O}_X) \xrightarrow{\exp} H^0(\mathcal{O}_X^*) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to ...$$

Definition 4.10 (Complexes, exact sequences of sheaves and resolutions): *A sequence*

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is called exact if $\ker(\psi) = \mathrm{im}(\phi)$.

A sequence

$$\ldots \to \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \ldots$$

is called a complex if $d^{i+1} \circ d^i = 0, \forall i \in \mathbb{Z}$.

Finally, a resolution of a sheaf $\mathcal{F}$ *is an exact comples of sheaves* $\{\mathcal{F}^i, d^i, i \in \mathbb{N}\}$ *with* $\ker d^0 \simeq \mathcal{F}$.

4.4 Examples of resolutions

The De Rham resolution Let X be a C^∞ manifold and let $\mathcal{A}^k(X)$ be the sheaf of sections of the k-the exterior power of the cotangent bundle, i.e. the sheaf of differential k-forms. Then the exterior derivative gives us a complex

$$0 \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \ldots$$

where $\mathcal{A}^0$ is the sheaf of smooth functions on X. The kernel of d^0 is precisely the constant functions, i.e. the constant sheaf $\underline{\mathbb{R}}$, and the Poincare lemma shows us that

$$\mathcal{A}^k \xrightarrow{d} \mathcal{A}^{k+1} \xrightarrow{d} \mathcal{A}^{k+2}$$

is locally exact, i.e. the sequence is an exact sequence of sheaves. Hence the De Rham complex is a resolution of the constant sheaf $\underline{\mathbb{R}}$.

The Dolbeault resolution In a similar way, when X is a complex manifold and E a holomorphic vector bundle, we showed in 2.4 that a Poincare-type lemma holds for the sheaf of smooth sections of the bundle $\mathcal{A}^{0,q}(E) = C^\infty(-, \Omega_X^{0,q} \otimes E)$ and hence we have a resolution

$$0 \to \mathcal{A}^{0,0}(E) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1}(E) \xrightarrow{\overline{\partial}} \ldots$$

of the sheaf $\mathcal{E}$ of holomorphic sections of E.

4.5 A bit of homological algebra

Our ultimate aim is to calculate the right derived functors of the left exact global sections functor $\mathrm{Sheaves}_X \to \mathrm{Ab}$. In our remarks on kernels and images, we have (almost) shown that the category of sheaves on X is an abelian category, which also has enough injective objects.

In such a context, to calculate right derived functors, one takes an injective resolution of the chosen object - these resolutions have the property that any two are related by a chain homotopy, and moreover if $\phi : A \to B$ is a map and $A \to I^\cdot, B \to J^\cdot$ are injective resolutions, there is a unique (up to chain homotopy) extension of ϕ to a map of complexes $I^\cdot \to J^\cdot$[5]. We have the following theorem:

Theorem 4.11 (Calculating right derived functors): *There is a universal δ-functor R^iF with the property that $R^0F = F$ and to every SES*

$$0 \to A \to B \to C \to 0$$

we have a LES

$$0 \to FA \to FB \to FC \to R^1FA \to R^1FB \to R^1FC \to ...$$

Proof. For a full proof, see Voisin or Weibel. The idea is the following: by the preceding remarks, $R^iF(A) = H^i(FI^\cdot)$, is well-defined. The idea then is to take $A \to I^\cdot, C \to J^\cdot$ and show that $B \to (I \oplus J)^\cdot$ is an injective resolution. Applying the additive functor F and using the snake lemma after taking cohomology provides us with the result. □

So we can compute the right derived functors using injective resolutions. However, injective resolutions are usually hard to find, and in fact we can replace them with arbitrary acyclic objects, where an object M is acyclic if $R^iF(M) = 0$ for $i > 0$ (from the construction, we immediately see that injectives are acyclic).

Theorem 4.12 (Derived functors using acyclic objects): *Right derived functors can be calculated using acyclic resolutions $A \to M^\cdot$, i.e. $R^iF(A) = H^i(FM^\cdot)$.*

Proof. The idea is to use dimension shifting. □

4.6 Sheaf cohomology

Now we are ready to define sheaf cohomology. The left exact functor we are interested in is the global sections functor, and we can calculate it using acyclic resolutions of a sheaf. Note that the category of sheaves on X has enough injectives, since the category of abelian groups has enough injectives and we can embed $\mathcal{F}$ into the sheaf $U \mapsto \bigoplus I_x$, where I_x is an injective group containing $\mathcal{F}_x$.

Definition 4.13 (Sheaf cohomology): *The sheaf cohomology $H^i(X, \mathcal{F})$ is defined as the i-th derived functor of the global sections functor Γ.*

Flasque and fine sheaves: There are two important classes of acyclic sheaves: one consists of the flasque sheaves, i.e. the sheaves such that the restriction maps are surjective. Another class

[5] For more on the homological algebra, consult Weibel

is the fine sheaves, i.e. the sheaves which admit a partition of unity. For a proof that these are acyclic, see Voisin, section 4.3.1

In particular, for the case of differentiable manifolds, where we have partitions of unity, the sheaf cohomology of the constant sheaf $\underline{\mathbb{R}}$ computes the De Rham cohomology:

Proposition 4.14 (De Rham cohomology): *The De Rham resolution computes the De Rham cohomology, i.e.* $H^i(X, \underline{\mathbb{R}}) = H^i_{DR}(X, \mathbb{R})$.

Similarly, the Dolbeault resolution computes the Dolbeault cohomology, as we would expect.

5 Hodge theory of Kähler manifolds

We conclude with a section proving the Hodge decomposition for compact Kähler manifolds. To do this, we first define the Hodge star operator, which will allow us to construct duals, or adjoints, of the operators d, ∂ and $\overline{\partial}$. We then define Laplacians and harmonic forms, and show that any cohomology class can be represented uniquely by an element of the vector space of harmonic forms. Then, we prove the Kähler identities, which allow us to show that the harmonic k-forms split into a sum of the harmonic $(p,q)-$forms, and then we conclude by using the isomorphism between k-th cohomology and the k-th harmonic forms.

5.1 The Hodge star and adjoints on smooth manifolds

Let X be a compact smooth manifold with a Riemannian metric. This induces a metric on the differential forms as follows: if $e_1, ..., e_n$ is an orthonormal basis for $T_{X,x}$, then the $e_{i_1}^* \wedge ... \wedge e_{i_k}^*$ form an orthonormal basis for $\Omega_{X,x}^k$.

Definition 5.1 (Hodge star): *The Hodge $*$ operator is the unique operator $\Omega_X^k \to \Omega_X^{n-k}$ such that*

$$\alpha \wedge *\beta = (\alpha, \beta)\mathrm{Vol}$$

where $\alpha, \beta \in \mathcal{A}^k(X)$ are sections of Ω_X^k and $*$ is induced from the operator on bundles given by composing a section with $*$.

The existence of the Hodge star operator is guaranteed by the following reasoning:

Firstly, we have the isomorphism

$$\Omega_{X,x}^{n-k} \simeq \mathrm{Hom}(\Omega_{X,x}^k, \Omega_{X,x}^n)$$

given by the right wedge product. This is an isomorphism, as the map is clearly injective and also the two vector spaces have the same dimension. Note that when the manifold is Riemannian, it has a volume form so $\Omega_{X,x}^n$ is canonically isomorphic to $\mathbb{R}$. Moreover, the metric gives us an isomorphism

$$\Omega_{X,x}^k \simeq \mathrm{Hom}(\Omega_{X,x}^k, \mathbb{R})$$

given by $\omega \mapsto (-,\omega)$. Composing these isomorphism we have:

$$\Omega_{X,x}^k \simeq \mathrm{Hom}(\Omega_{X,x}^k, \mathbb{R}) \simeq \mathrm{Hom}(\Omega_{X,x}^k, \Omega_{X,x}^n) \simeq \Omega_{X,x}^{n-k}$$

Denoting this composite map $*$ and unraveling the definitions, we see that for a section $\beta \in \mathcal{A}^k(X)$, $*\beta$ is the element in $\mathcal{A}^{n-k}(X)$ such that wedging with it produces the same map as using the metric:

$$- \wedge *\beta = (-, \beta)\mathrm{Vol}$$

.

Definition 5.2 (L^2 metric): *On elements $\alpha, \beta \in \mathcal{A}^k(X)$ we have the L^2 metric defined by*

$$(\alpha, \beta)_{L^2} = \int_X (\alpha, \beta)\mathrm{Vol},$$

where $x \mapsto (\alpha_x, \beta_x)_x$ is a function of x.

Immediately from the definition we see that $(\alpha, \beta)_{L^2} = \int_X \alpha \wedge *\beta$.

Proposition 5.3: *The Hodge star operator satisfies $*^2 = (-1)^{k(n-k)}$.*

Proof. $*$ preserves metrics, so we have

$$\alpha_x \wedge *\beta_x = (\alpha_x, \beta_x)_x \mathrm{Vol}_x = (*\alpha_x, *\beta_x)_x \mathrm{Vol}_x = *\beta_x \wedge * * \alpha_x = (-1)^{k(n-k)} * *\alpha_x \wedge *\beta_x$$

□

Let $d : \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)$ be the exterior derivative and define $d^* = (-1)^k *^{-1} \circ d \circ * = (-1)^{n(k+1)+1} * \circ d \circ *$. This is called the adjoint to d for the following reason:

Proposition 5.4 (Adjoint property): *If X is compact or only compactly supported integration is allowed, then*

$$(\alpha, d^*\beta)_{L^2} = (d\alpha, \beta)_{L^2}$$

Proof. Let $\alpha \in \mathcal{A}^{k-1}(X), \beta \in \mathcal{A}^k(X)$. Then $(d\alpha, \beta)_{L^2} = \int_X d\alpha \wedge *\beta$. However, $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{k-1}\alpha \wedge d * \beta$. Integrating over X and using Stokes' theorem, we get

$$(d\alpha, \beta)_{L^2} = (-1)^k \int_X \alpha \wedge d * \beta$$

But $(\alpha, d^*\beta)_{L^2} = \int_X \alpha \wedge *(-1)^k *^{-1} d * \beta = (-1)^k \int_X \alpha \wedge d * \beta$ and so the two quantities are equal. □

5.2 The operators ∂ and $\overline{\partial}$ on complex manifolds

The Hodge star operator was defined for smooth manifolds in the previous section. Now let X be a compact complex manifold. We can extend the Riemannian metric to a Hermitian metric on the complexified cotangent bundle and extend $*$ $\mathbb{C}$-linearly to complex-valued forms. In local coordinates, if

$$\alpha = \sum \alpha_{I,J} dz_I \wedge d\overline{z}_J, \beta = \sum \beta_{I,J} dz_I \wedge d\overline{z}_J$$

are in $\Omega_X^{p,q}$, then their Hermitian product at x is equal to

$$(\alpha_x, \beta_x)_x = \sum \alpha_{I,J}(x)\overline{\beta_{I,J}(x)}$$

We then have the identity

$$(\alpha_x, \beta_x)\text{Vol}_x = \alpha_x \wedge \overline{*\beta_x}$$

and the Hodge star takes a (p, q) form to an $(n-p, n-q)$ form.

Recall that $d = \partial + \overline{\partial}$ on complex manifolds and since a complex manifold has underlying even dimension, then $(-1)^{n(k+1)+1} = -1$ and we can define the duals of ∂ and $\overline{\partial}$ to be $\partial^* = - * \partial *, \overline{\partial}^* = - * \overline{\partial} *$. These satisfy the same adjoint property as d:

Proposition 5.5: *∂^* and $\overline{\partial}^*$ are formal adjoints of ∂ and ∂^* respectively.*

Proof. We show this for ∂, the other case being analogous. Firstly, if α is of type $(p-1, q)$ and β of type (p, q) with $n = p + q$ beng the dimension of X as a complex manifold, then

$$(\partial\alpha, \beta)_{L^2} = \int_X \partial\alpha \wedge \overline{*\beta}$$

However, $\alpha \wedge \overline{*\beta}$ is of type $(n-1, n)$ and hence $\partial(\alpha \wedge \overline{*\beta}) = d(\alpha \wedge \overline{*\beta})$. Now by Stokes' theorem,

$$0 = \int_X d(\alpha \wedge \overline{*\beta}) = \int_X \partial(\alpha \wedge \overline{*\beta}) = (\partial\alpha, \beta)_{L^2} + (-1)^{p+q-1} \int_X \alpha \wedge \partial\overline{*\beta} \tag{5.1}$$

But note that

$$(\alpha, \partial^*\beta)_{L^2} = \int_X \alpha \wedge *\overline{\partial^*\beta} = \int_X \alpha \wedge *\overline{- * \partial * \beta} = (-1) \int_X \alpha \wedge \overline{* * \partial * \beta}$$

The last equality comes from the fact that $*$ is a real operator. But $\partial * \beta$ is a form of type $(n-p+1, n-q)$ on which $**$ acts as $(-1)^{2n-p-q+1} = (-1)^{p+q+1}$ and hence

$$(\alpha, \partial^*\beta)_{L^2} = (-1)^{p+q} \int_X \alpha \wedge \partial\overline{*\beta} \tag{5.2}$$

Combining (6.1) and (6.2) gives the result. □

Remark: The preceding constructions can also be extended to the case of holomorphic vector bundles with the operator $\overline{\partial}_E$ as in the Dolbeault complex of a holomorphic vector bundle 2.5.

5.3 Laplacians, harmonic forms and cohomology

For any differential operator θ, e.g. d, ∂ or $\overline{\partial}$ define its associated Laplacian as

$$\Delta_\theta = \theta\theta^* + \theta^*\theta$$

As a corollary of the adjunction properties 5.4 and 5.5 we have:

Corollary 5.6: $(\alpha, \Delta_\theta\beta)_{L^2} = (\theta\alpha, \theta\beta)_{L^2} + (\theta^*\alpha, \theta^*\beta)_{L^2}$. *In particular,* $(\alpha, \Delta_\theta\alpha)_{L^2} = ||\theta\alpha||^2 + ||\theta^*\alpha||^2$.

Definition 5.7 (Harmonic forms): *A θ-harmonic form is a form α such that $\Delta_\theta \alpha = 0$*

Hence, by applying 5.6, we see that a form is θ-harmonic if and only if it is θ and θ^*-closed:

Corollary 5.8: $\ker \Delta_\theta = \ker \theta \cap \ker \theta^*$

Definition 5.9 (Vector space of harmonic forms): *Define $\mathcal{H}_d^k$ (resp. $\mathcal{H}_{\overline{\partial}}^k$) to be the space of all d (resp. $\overline{\partial}$)-harmonic forms, and $\mathcal{H}_d^{p,q}$ (resp. $\mathcal{H}_{\overline{\partial}}^{p,q}$) the d (resp. $\overline{\partial}$)-harmonic forms of type (p,q).*

Now we show that the De Rham cohomology groups are isomorphic to these harmonic vector spaces, using a big theorem about elliptic differential operators which we quote without proof:

Theorem 5.10 (Big theorem on elliptic differential operators): *Let $P : E \to F$ be an EDO on a compact manifold. If E and F are of the same rank and are equipped with metrics, then $\ker P$ is of finite dimension and there is an L^2 orthogonal decomposition*

$$C^\infty(E) = \ker P \oplus P^*(C^\infty(F),$$

where P^* is the formal adjoint of P.

We will apply this to the Laplacian Δ_d, which is an elliptic differential operator of degree 2, which is also self-adjoint: $\Delta = \Delta^*$. In particular, we have

$$\mathcal{A}^k(X) = \mathcal{H}^k \oplus \Delta(\mathcal{A}^k(X))$$

Now let's see what happens when we pass to cohomology: let β be a closed form, $\beta = \alpha + \Delta\gamma$ with α harmonic, i.e. $\beta = \alpha + dd^*\gamma + d^*d\gamma$. But now β, α and $dd^*\gamma$ are all closed, hence $d^*d\gamma$ is closed, $d^*d\gamma \in \ker d \cap \operatorname{im} d^*$. But $0 \leq (d^*d\gamma, d^*d\gamma) = (d\gamma, dd^*d\gamma) = 0$ and hence $d^*d\gamma = 0$. Hence β is represented by a harmonic form modulo some exact form, and the map $\mathcal{H}^k \to H^k(X)$ is surjective. Conversely, to show injectivity, assume β is harmonic and exact. Then $\beta \in \ker d^* \cap \operatorname{im} d$ and again it must be 0. We conclude that:

Theorem 5.11: *Let X be a compact oriented Riemannian manifold. Then the map*

$$\mathcal{H}^k \to H^k(X)$$

is an isomorphism

Remark: Note that this statement applies to the real-valued De Rham cohomology when dealing with the usual exterior derivative, but also works when we extend it $\mathbb{C}$-linearly with complex-valued De Rham cohomology.

We can apply the same idea to the Laplacian associated to $\overline{\partial}$ to get the decomposition

$$C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,q}))$$

Using the exact same reasoning, we get the following:

Theorem 5.12: *Let X be a compact complex manifold with a Hermitian metric. Then the map*

$$\mathcal{H}^{p,q} \to H^{p,q}(X)$$

is an isomorphism. In particular, the Dolbeault cohomology groups have finite dimension.

5.4 The case of Kähler manifolds

Our aim now is to use the isomorphism between the harmonic and ordinary cohomology groups, together with the decomposition $\mathcal{H}^k = \bigoplus \mathcal{H}^{p,q}$ to prove the Hodge decomposition theorem, the final theorem in this note.

To do this, we will work entirely with compact Kähler manifolds (the decomposition theorem does not necessarily hold for non-Kähler manifolds) and prove the so-called Kähler identities to establish the equality between the different Laplacians acting on X.

5.4.1 The Kähler identities

Definition 5.13 (Lefschetz operator): *Define the Lefschetz operator on complex differential forms*

$$L : \mathcal{A}_X^k \to \mathcal{A}_X^{k+2}$$

by $\alpha \mapsto \omega \wedge \alpha$, where ω is the Kähler form. Its formal dual is

$$\Lambda : \mathcal{A}_X^k \to \mathcal{A}_X^{k-2}$$

where $\Lambda = (-1)^k * L*$

The construction of the adjoint can be verified by seeing that

$$\alpha \wedge *\Lambda\beta = (\alpha, \Lambda\beta)\mathrm{Vol} = (L\alpha, \beta)\mathrm{Vol} = L\alpha \wedge *\beta = \alpha \wedge \omega \wedge *\beta$$

i.e. $*\Lambda = L*$, or $\Lambda = *^{-1}L*$.

Proposition 5.14 (Kähler identities):

$$[\Lambda, \overline{\partial}] = -i\partial^*, [\Lambda, \partial] = i\overline{\partial}^*$$

Proof. See Voisin, section 6.1. □

Corollary 5.15 (Comparing the Laplacians): *We have $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$*

Proof.

$$\Delta_d = dd^* + d^*d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \tag{5.3}$$

Notice that by the Kähler identities,

$$\partial^*\overline{\partial} = i[\Lambda, \overline{\partial}]\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial}$$

and similarly

$$\overline{\partial}\partial^* = i\overline{\partial}\Lambda\overline{\partial}$$

i.e.

$$\partial^*\overline{\partial} = -\overline{\partial}\partial^*$$

Also, note that we have $\partial\overline{\partial} = -\overline{\partial}\partial$.

Expanding (6.3) we get

$$\Delta_d = \partial\partial^* + \partial\overline{\partial}^* + \overline{\partial}\partial^* + \overline{\partial}\overline{\partial}^* + \partial^*\partial + \partial^*\overline{\partial} + \overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial}$$

Now, the gray bits are both 0 and we are left with

$$\Delta_d = \Delta_\partial + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

But $\overline{\partial}^* = -i[\Lambda, \partial]$ so we get

$$\Delta_d = \Delta_\partial + \overline{\partial}(-i\Lambda\partial + i\partial\Lambda) + (-i\Lambda\partial + i\partial\Lambda)\overline{\partial} = \Delta_\partial + i\partial[\Lambda, \overline{\partial}] + i[\Lambda, \overline{\partial}]\partial = 2\Delta_\partial$$

The other case is proved in exactly the same way. □

Now, since Δ_∂ is bihomogenous, i.e. keeps the bigrading the same, the same will apply to Δ_d. Hence, if we have a d-harmonic form $\alpha = \sum \alpha^{p,q}$, we deduce that each $\alpha^{p,q}$ is d-harmonic. In other words,

Theorem 5.16:

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

Notice that $\mathcal{H}^{p,q} = \overline{\mathcal{H}^{q,p}}$ since if β is harmonic of type (p,q), then $\overline{\beta}$ is of type (q,p) and

$$\overline{\Delta_\partial \overline{\beta}} = \Delta_{\overline{\partial}}\beta = \Delta_\partial\beta = 0$$

i.e. $\overline{\beta}$ is harmonic as well.

Now recall that by theorem 5.11 we have that

$$\mathcal{H}^{p,q} \simeq H^{p,q}(X)$$

and

$$\mathcal{H}^k \simeq H^k(X)$$

This allows us to conclude:

Theorem 5.17 (Hodge decomposition): *We have the decomposition*

$$H^k(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

References

[1] Weibel, C. *An Introduction to Homological Algebra* Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, Cambridge, 1994.

[2] Huybrechts, D. *Complex Geometry*

[3] Voisin, C. *Hodge Theory and Complex Algebraic Geometry, I*

[4] Horing, A. *Kähler Geometry and Hodge theory*

[5] Demailly, Jean-Pierre. *Complex Analytic and Differential Geometry*