Introduction to Lagrangian torus fibrations and SYZ mirror symmetry

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Overview

- Lagrangian torus fibrations and the Arnold-Liouville theorem
- SYZ mirror symmetry and moduli of special Lagrangians
- Examples
- General AAK construction (if time permits)

The Arnold-Liouville theorem

Suppose we have an integrable system

$$H : X \to \mathbb{R}^n$$

We call this a Hamiltonian system if

$$\{H_i, H_j\} = 0$$

This implies but is stronger than the usual Frobenius integrability of the Hamiltonian vector fields, i.e. $[V_i, V_j] = 0$. We assume their flows are defined for all time and hence produce an $\mathbb{R}^n$ action on X:

$$t \cdot x = \phi_{t_1}^{H_1} \dots \phi_{t_n}^{H_n}(x)$$

Proposition : *The orbits of the $\mathbb{R}^n$ action are isotropic. Moreover, if $n = \frac{1}{2}\dim(X)$, then the connected regular fibers of H are Lagrangian orbits.*

The Arnold-Liouville theorem

We also require that H is proper and the base contains a dense open set of regular values. In this case, we have the following:

Theorem (Arnold-Liouville): *If $H : X \to \mathbb{R}^n$ is an integrable Hamiltonian system, then any regular fibre is a Lagrangian torus admitting a neighbourhood symplectomorphic to $B \times T^n, \omega = \sum db_i \wedge dt_i$ with the standard torus action.*

This is saying that the $\mathbb{R}^n$ action descends to a local T^n action. When there is a global torus action, we are in the realm of *moment maps* and *symplectic toric varieties*. The first part of the theorem is easy (look at stabilizer at a point) , but the second one requires the use of *action-angle* coordinates. The action coordinates are given by the *flux map*:

$$I(b) = \underbrace{(\frac{1}{2\pi}\int_{\gamma_i}\lambda)_{i=1}^n}_{\omega = d\lambda \text{exact}} = (\frac{1}{2\pi}\int_{\Gamma_i}\omega)_{i=1}^n$$

Example: the Auroux system

We take the Hamiltonians

$$H : \mathbb{C}^2 \to \mathbb{R}^2$$

$$H_1(z_1, z_2) = |z_1 z_2 - c|^2, H_2(z_1, z_2) = \frac{1}{2}(|z_1|^2 - |z_2|^2)$$

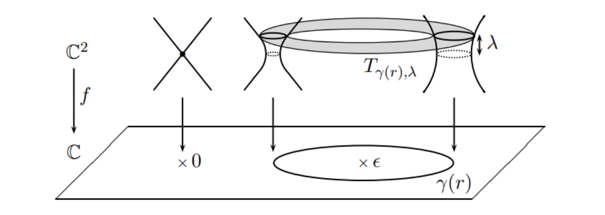


Figure 1: Figure from Auroux's paper on T-duality; $f = z_1 z_2$

Example: the Auroux system

This example has a singularity at the origin, giving rise to a pinched torus fibre which in turn produces monodromy, a Dehn twist!

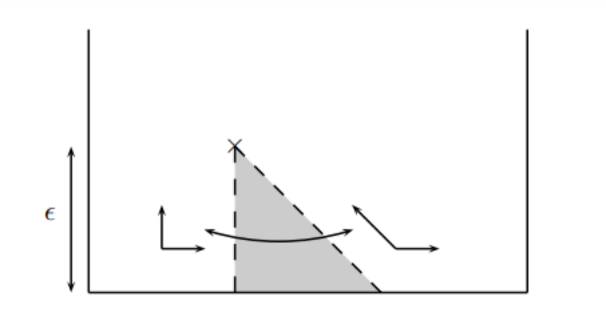
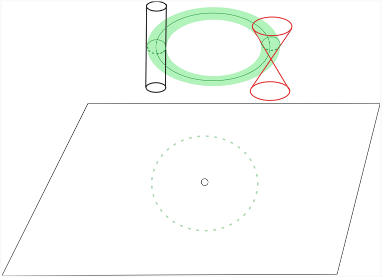


Figure 2: Affine shear

The Auroux system: action-angle coordinates



Let us see the action-angle coordinates producing an integral affine structure on the base.

- For small radius and $\lambda > 0$, we have a section over the base, denoted β and a thimble associated to the height α.
- For large radius, we have two sections over the base, which we denote β_0, β_1

We have that $\alpha = \beta_0 - \beta_1$. However, β transforms to β_1 when $\lambda > 0$ and β_0 when $\lambda < 0$. Clockwise monodromy is $(\alpha, \beta) \mapsto (\alpha, \beta - \alpha)$!

Note that the coordinate α is unchanged, as it comes from a global S^1 action of which H_2 is the moment map.

Mirror symmetry for Lagrangian torus fibrations

HMS heuristically produces Lagrangian mirrors to sheaves. Taking a skyscraper sheaf, we see that we must have

$$Ext(\mathcal{O}_p, \mathcal{O}_p) \simeq H^\bullet(T^n, \mathbb{C}) \simeq HF(L, L)$$

So naturally, we expect the mirrors to skyscraper sheaves to be Lagrangian tori and hence roughly

$$X = \text{moduli of skyscraper sheaves} = \text{moduli of Lagrangian tori in } X^\vee$$

We will see how this philosophy gives a recipe, called SYZ mirror symmetry, of producing mirror pairs by constructing the moduli of Lagrangian tori as a family in a fibration.

Moduli of special Lagrangians and McLean's theorem

Recall from Emily' talk: a deformation $\nu \in H^0(L, \mathcal{N}_L)$ is Lagrangian if

$$0 = \mathcal{L}_\nu \omega = d\iota_\nu \omega$$

and is special if

$$0 = \mathcal{L}_\nu \mathrm{im}\Omega = d\iota_\nu \mathrm{im}\Omega$$

If we put $\alpha = -\iota_\nu \omega, \beta = \iota_\nu \mathrm{im}\Omega$, we in fact have a relationship $\beta = \psi \star_g \alpha$.

Theorem (McLean): *The deformations of a special Lagrangian are controlled by the ψ-harmonic forms, which by Hodge theory correspond to $H^1(L; \mathbb{R})$:*

$$\{\textit{special Lagrangian deformations of } L\} \leftrightarrow \mathcal{H}^1_\psi(L) \simeq H^1(L; \mathbb{R})$$

If we are given a torus fibration $X^0 \to B$, this identifies

$$T_bB \simeq H^1(p^{-1}(b), \mathbb{R}) \simeq H^{n-1}(p^{-1}(b), \mathbb{R}), \nu \mapsto -\iota_\nu \omega \text{ or } \nu \mapsto \iota_\nu \mathrm{im}\Omega$$

Moduli of special Lagrangians: complex structure

We complexify the moduli space of special Lagrangians B by equipping them with local systems $\nabla = d + iA$. This space $\mathcal{M}$ carries a Kahler structure:

$$T_{L,\nabla}\mathcal{M} = \{(\nu, A) \in H^0(L, \mathcal{N}_{L/X}) \times \Omega^1(L)\} \simeq \mathcal{H}^1_\psi(L) \otimes \mathbb{C}$$

$$(\nu, A) \mapsto iA - \iota_\nu \omega$$

$$J(\nu, A) = (\nu', A'), A' = -\iota_\nu \omega, \iota_{\nu'}\omega = A \qquad \text{(complex structure)}$$

$$\Omega^\vee_{L,\nabla}(\nu_j, A_j) := \int_L \bigwedge_j iA_j - \iota_{\nu_j}\omega \qquad \text{(holomorphic n-form)}$$

$$\omega^\vee_{L,\nabla}((\nu_1, A_1), (\nu_2, A_2)) := \int_L A_2 \wedge \iota_{\nu_1} Im\Omega - A_1 \wedge \iota_{\nu_2} Im\Omega \quad \text{(Kahler form)}$$

The symplectic and holomorphic structures are exchanged!

Moduli of special Lagrangians: coordinates

We have a pairing

$$T_bB \times H_1(p^{-1}(b)) \to \mathbb{R}$$
$$(\nu, \gamma) \mapsto \int_\gamma -\iota_\nu \omega = -\int_\Gamma \omega$$

Via the exponential map and after choosing a basis of H_1 this gets us local real coordinates on B. Similarly, after complexifying we obtain local complex coordinates for $\mathcal{M}$:

$$(L, \nabla) \mapsto \left(\exp\left(-\int_{\Gamma_i} \omega\right) \mathrm{hol}_\nabla(\partial\gamma_i)\right)_i$$

We can also define holomorphic functions for any $\beta \in H_2(X, L)$:

$$z_\beta : \mathcal{M} \to \mathbb{C}^\times$$
$$z_\beta(L, \nabla) = \exp(-\int_\beta \omega)\mathrm{hol}_\nabla(\partial\beta)$$

The superpotential as curvature

In Floer theory, one considers chain complexes

$$CF(L, L') = \bigoplus_{p \in L \cap L'} \Lambda \cdot p, dp = \sum_{[u]=1} \mathcal{M}(p, q; [u])q$$

The differential in this complex squares to zero in favorable situations. The obstruction comes in the form of a curvature term

$$d^2 = (W(L) - W(L'))id$$

which counts index 2 disks with boundary on either L or L'- notice how different this is from Morse theory.

Formally,

$$W(L, \nabla) := \sum_{\mu(\beta)=2} n_\beta(L) z_\beta : \mathcal{M} \to \mathbb{C}$$

The point is that d^2 has contributions coming from the boundary of the moduli space of index 2 strips, which is a compact 1-manifold:

$$\partial\mathcal{M}(p,q;[u]) = \coprod_{[u]=[u']+[u'']} \mathcal{M}(p,q;[u']) \times \mathcal{M}(p,q;[u''])$$

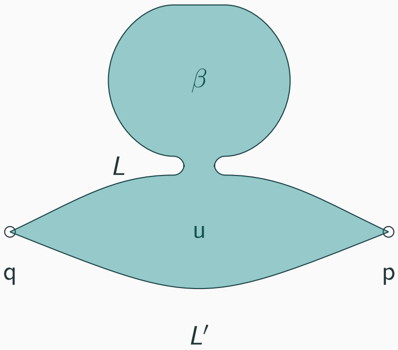
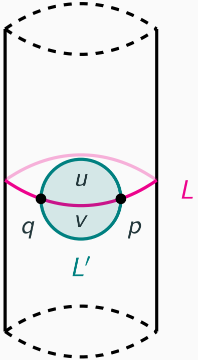


Figure 3: Bubbled degeneration whose limit contributes to $W(L)$ term in d^2

Example



$$W(L) = 0$$

$$W(L') = z_\beta = \exp(-\int_\beta \omega) = \exp(-\int_u \omega)\exp(-\int_v \omega)$$

$$dp = \pm\exp(-\int_u \omega)q$$

$$dq = \pm\exp(-\int_v \omega)p$$

$$d^2 = \pm W(L')id$$

Wall-crossing and scattering diagrams

We see that we can do Floer cohomology only when $W(L) = W(L')$. This divides the Lagrangian tori into chambers $U^\vee$, divided by walls of *potentially obstructed* Lagrangian tori (bounding Maslov 2 disks). To construct a mirror space to a torus fibration $X^0 \to B$, one has to glue the chambers across the walls - the correction maps are called *instanton corrections*.

The toric case: no wall-crossing

Toric varieties are compactifications of $(\mathbb{C}^\times)^n$ which comes with the standard torus fibration

$$(\mathbb{C}^\times)^n \to \mathbb{R}^n$$

All the tori are standard and unobstructed, so the mirror is just $(\mathbb{C}^\times)^n$ again. Adding in the toric boundary divisor, we need to modify the mirror by a superpotential accounting for the new Maslov 2 disks hitting the boundary. For example,

$$\mathbb{C}^\times \leftrightarrow \mathbb{C}^\times$$

$$\mathbb{CP}^1 \leftrightarrow (\mathbb{C}^\times, W = z + \frac{1}{z})$$

The Auroux system again

$$X^0 = \mathbb{CP}^2 \setminus \text{line} \cup \text{conic} = X \setminus D = \mathbb{C}^2 \setminus \{xy = c\}$$

We construct the mirror of $\mathbb{C}^2$ relative to $D = \{xy = c\}$, a generic fiber: The Lagrangian torus fibration on X^0 has two chambers:

- Small radius chamber: Here, the fibration is trivial over a punctured disk, so admits no holomorphic disks bounding the torus. Once we add in the divisor, we fill in the puncture and get a section which is the only holomorphic disk appearing in the superpotential. In coordinates (w, u) associated to (α, β) we have

$$W = u$$

The Auroux system again

- Large radius chamber: here, we can isotope the tori to be centered at the origin and become standard product tori in $(\mathbb{C}^\times)^2$. Hence, once we add the boundary divisor this reduces to the toric case in the toric variety $\mathbb{C}^2$ which has two boundary components giving rise to two holomorphic disks β_0, β_1. We can define coordinates $(w, v) \leftrightarrow (\alpha, -\beta_1)$, which express the superpotential as

$$W = z_{\beta_0} + z_{\beta_1} = \frac{1}{v} + \frac{w}{v}$$

The class β deforms to both β_0 and $\beta_1 = \beta_0 + \alpha$ due to bubbling once we cross the singular wall. Identifying

$$u = z_\beta = z_{\beta_0} + z_{\beta_1}$$

results in an *instanton correction* making the superpotential globally defined:

$$W = u = \frac{w+1}{v} \implies uv = w + 1$$

The Auroux system again

The example is self-mirror:

$$\mathbb{C}^2 \setminus D \leftrightarrow \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^\times \mid uv = w + 1\}$$

In fact, Pascaleff in his thesis explores this example and finds a Lagrangian section of the SYZ fibration L such that

$$SH^0 = HW(L, L) = \mathbb{C}[u, v, w][(uv - 1)^{-1}] = \mathrm{Ext}(\mathcal{O}, \mathcal{O})$$

Remark (Superpotential trickery): *In this example, we used a bit of trickery: we started with the open Calabi-Yau $X^0 = \mathbb{C}^2 \setminus D$ and constructed a torus fibration. A priori, we didn't know how to glue the two charts together. Initially, there are no Maslov 2 disks bounding the torus fibers. However, by partially compactifying to $\mathbb{C}^2$, we get a superpotential which is different in both chambers (the Chekanov and Clifford tori). The trick is that the gluing maps must identify the superpotential, so computing it in both chambers tells us enough to understand them!*

Another example with a single wall-crossing

In a similar way, we can construct the mirror to $\mathcal{O}(-2) = K_{\mathbb{P}^1}$ which we think of as a degeneration of $\mathbb{C}^\times$ to $\mathbb{A}^1 \cup \mathbb{P}^1 \cup \mathbb{A}^1$.

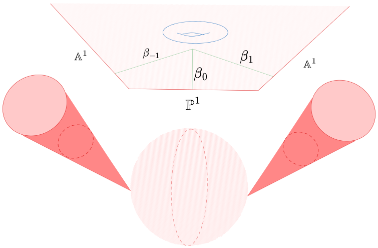


Figure 4: Toric boundary

- Small radius chamber: trivial fibration, $W = u$.
- Large radius chamber: toric case, $W = \frac{w+1+w^{-1}}{v}$

Another example with a single wall-crossing

We need to equate

$$W = u = \frac{w + 1 + w^{-1}}{v}$$

Hence

$$K_{\mathbb{P}^1} \leftrightarrow (X^0 = \{(w, u, v) \in \mathbb{C}^\times \times \mathbb{C}^2 |\, uv = f(w)\}, W = u)$$

where $f(w) = w + 1 + w^{-1}$. This has only two singular fibres at the roots of f. If we were to remove a generic fibre from $K_{\mathbb{P}^1}$, this would remove the superpotential from the mirror X^0. In fact, if we partially compactify X^0 by allowing $w \in \mathbb{C}$, we get the affine quadric $X \simeq T^*S^2$.

Hyperkahler rotations are not always self-mirror

The previous example shows that T^*S^2 is almost mirror to its hyperKahler rotation $K_{\mathbb{P}^1}$, but not quite! One needs to insert a superpotential and delete a generic fiber resulting in an LG model $(Y^0 = K_{\mathbb{P}^1} \setminus \mathbb{C}^\times, W)$

Proposition (Mirror symmetry for T^*S^2): *There is a derived equivalence*

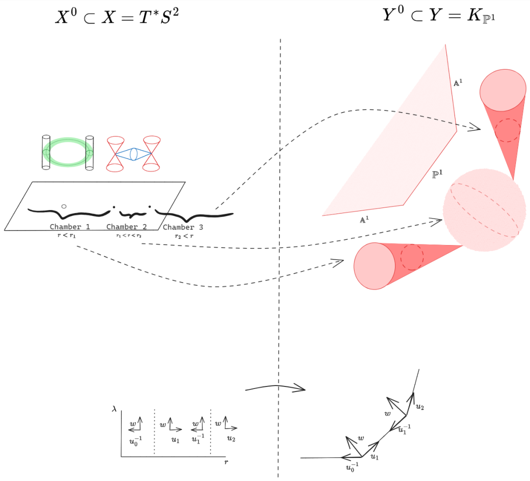
$$\mathcal{W}(T^*S^2) \simeq D^b(Y^0, W) \simeq D^b(\mathbb{C}[x]/x^2)$$

sending the cotangent fiber which generates the wrapped Fukaya category to the generator of the matrix factorizations and the zero section which generates the compact Fukaya category to the generator of $\mathrm{Perf}(\mathbb{C}[x]/x)$

Note that HMS would trivially not hold for T^*S^2 and $K_{\mathbb{P}^1}$ as the former has one generator, whereas the latter has two.

Big picture

We summarize the story in the following picture: there are three chambers of the SYZ fibration, roughly corresponding to the hyperplanes in the polytope of the toric mirror:



Base-fiber duality

Both of the previous examples, which are toric varieties and thought of as degenerations of $\mathbb{C}^\times$, were found to be mirror to certain conic fibrations. In fact, this is the approach AAK take. The idea is that the toric boundary encodes information about the singular locus of the conic fibration, a hypersurface in the base of the mirror. This phenomenon is called *base-fiber duality*, and the picture is as follows:

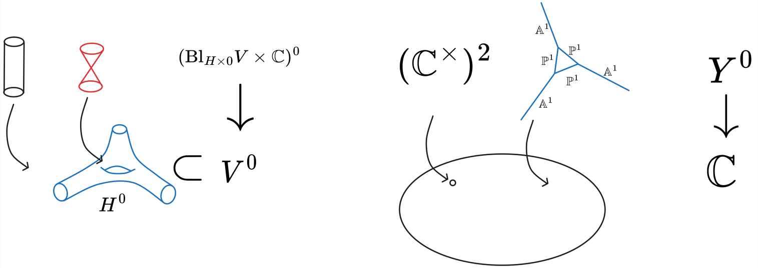


Figure 5: Base-fiber duality: the thrice-punctured elliptic curve is dual to the toric boundary of $K_{\mathbb{P}^2}$, which appears as the amoeba in the polytope

References

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Thank you for your attention!