# Symplectic cohomology and spectra

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# Overview

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- Introduction to symplectic cohomology

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- Finite dimensional approximations: Milnor's Morse theory of the energy functional

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- Finite dimensional approximations: Milnor's Morse theory of the energy functional
- A spectrum representing SH of cotangent bundles

# Symplectic cohomology

# Hamiltonian Floer cohomology

If $(M, d\theta)$ is an exact symplectic manifold and $H$ is a Hamiltonian function on it, we can study the infinite-dimensional Morse theory of the action functional

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- differential counting solutions to Floer's equation $\overline{\partial}_J u = \nabla H$
- The energy of a solution satisfies

$$0 \leqslant E(u) = \int |\partial_s u|^2 \mathrm{ds} \wedge \mathrm{dt} = \int \omega(\partial_s u, \partial_t u - X)\,\mathrm{ds} \wedge \mathrm{dt}$$

  which one should think of as roughly $E = \mathrm{d}\mathcal{A}$ and thus $E(u) = \mathcal{A}(\gamma-) - \mathcal{A}(\gamma_+)$. This means that the differential decreases the action.

# Comparison with quantum cohomology

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We will be interested in non-compact $M$, for example affine algebraic varieties, for which the analogue of quantum cohomology becomes *symplectic cohomology*.

# Liouville domains

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$$\hat{M} = M \cup_{\partial M} [0, \infty) \times \partial M$$

$$\theta|_{[0,\infty)\times\partial M} = e^r \alpha, Z|_{[0,\infty)\times\partial M} = \partial_r$$

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The result is called the completion of Liouville domain. More generally, a Liouville manifold is a manifold exhausted by Liouville domains.

# First definition of symplectic cohomology

Take a Hamiltonian on $\hat{M}$ which at the infinite end looks like $h(e^r)$, where $h : \mathbb{R} \to \mathbb{R}$ is a quadratic function. We define

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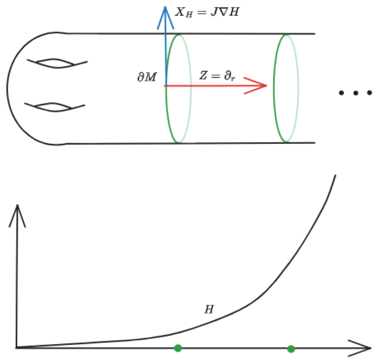
Since $X_H = h'(e^r)R, \nabla H = h'(e^r)\partial_r$, one can identify the 1-periodic orbits of $X_H$ with the Reeb orbits of arbitrary period on the boundary!



# Alternate definition

Alternatively, one could take a family of Hamiltonians $H^\tau$ which initially have increasing slope, but after a point look like $\tau e^r + c$. They only capture Reeb orbits of period $< \tau$. To capture all of the orbits, we take a direct limit:

$$SH^\bullet(M) := \varinjlim HF^\bullet(H^\tau)$$

# Properties of SH

There is a canonical map $H^\bullet(M) \to SH^\bullet(M)$ accounting for the low-energy solutions (where the Hamiltonian is close to zero, in the interior).

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Moreover, different moduli problems equip $SH^\bullet$ with a product and a BV operator.

# Example: surfaces

For example, in the punctured genus $g$ surface, the Morse-Bott spectral sequence has first page

$$E_1 = H^\bullet(M) \oplus \bigoplus_{i \geqslant 0} H^\bullet(S^1)[i(4g-2)]$$

since the Maslov index in this case is the Chern number of the surface, which is the Euler characteristic $\chi = 2 - 4g$. For $g > 1$ this degenerates for degree reasons!

# Example: Unit ball

$M = D^2, \hat{M} = \mathbb{C}$. One can show that this vanishes using either approach:

- If we choose $H = (\frac{1}{2}|x|^2)^2 + 2\pi k + \frac{1}{2}$, then we have one stationary point at 0, together with 1-periodic orbits at $\frac{1}{2}|x|^2 = \pi l, l > k$. The Conley indices live in degrees

  $$-n - 2nk, -n - 2nk - 1, -n - 2n(k+1), -n - 2n(k+1) - 1, \ldots$$

  By taking $k$ arbitrarily large, we see that $SH^\bullet(D^2) = 0$.

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  By taking $k$ arbitrarily large, we see that $SH^\bullet(D^2) = 0$.
- Equivalently, we could have taken a linear Hamiltonian $H = \tau_k \frac{1}{2}|x|^2$ for a sequence $\tau_k \in (2\pi k, 2\pi(k+1))$. Each of these has a single critical point at 0, and it can be shown that the continuation maps all vanish.

# Viterbo's theorem

**Theorem (Viterbo's theorem):** *Suppose $N$ is oriented, spin and closed. Then*

$$SH^\bullet(T^*N) \simeq H_{-\bullet}(\mathcal{L}N)$$

The idea is that periodic Reeb orbits on $T^*N$ correspond to closed geodesics, which are critical points of the energy functional on the free loop space.

# Ganatra's categorical results

It has been shown by Ganatra that the symplectic cohomology can be recovered from the wrapped Fukaya category:

**Theorem (Ganatra):**

$$\mathrm{HH}^\bullet(\mathcal{W}(M)) \simeq \mathrm{SH}^\bullet(M)$$
$$\mathrm{HC}^\bullet(\mathcal{W}(M)) \simeq \mathrm{SH}^\bullet_{S^1}(M)$$

# Finite-dimensional approximations

# Classical perspective

Milnor studied the Morse theory of the energy functional

$$E : \mathcal{L}N \to \mathbb{R}$$
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$$\mathcal{L}(\underline{t}) := \mathcal{L}(0, t_1, \ldots, t_k = 1)^{E \leqslant c} \to N^k$$
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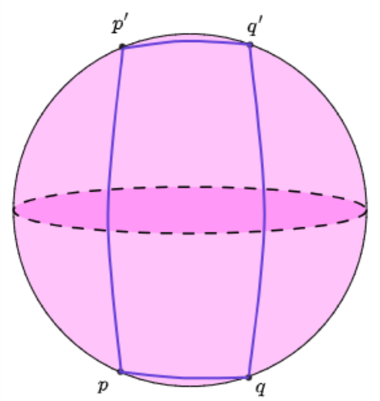
This is a deformation retract of the space of loops with energy less than a fixed $c$, and moreover has the homotopy type of a finite CW complex. Taking finer and finer subdivisions $\underline{t}$ of $S^1$, as well as higher energy, leads us closer to the loop space.

# Loop space of a sphere

We can understand $\Omega S^n \simeq \mathcal{P}(p, q; S^n)$ via the geodesics starting at $p$ and ending at $q$. There are infinitely many of them:

$$\gamma_0 = p, \gamma_1 = pp'q'q, \gamma_2 = pqq'p'pq, \ldots$$

which have index $0, n-1, 2(n-2), \ldots$. When $n > 2$ this exactly determines the homology of the based loop space!



# Towards the generating function spectrum

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Milnor tells us how to approximate the loop space of $N$: take the piecewise geodesics corresponding to a subdivision of the interval. Kragh just takes the cotangent bundle of Milnor's construction:

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# Conley indices

To make this promotion to spectra, we need to use so-called Conley indices.

**Definition (Conley indices):** *A pair of compact spaces $B \subset A \subset M$ is a Conley index pair for $f : M \to \mathbb{R}$ relative to $(a, b)$ if*

- $B \subset A \subset f^{-1}([a, b])$
- $\mathrm{int}(A \setminus B)$ *contains all critical points of $f$ with values in $(a, b)$.*
- *A flow line in $A$ either stays in $A \setminus B$ converging to a critical point, or exits through $B$.*

The quotient space, if the index pair is good, is denoted

$$I_a^b(f) := A/B$$

and has the homotopy type of a CW complex.

# Approximating the action functional

The finite-dimensional approximation to the action functional is encoded in the following picture:

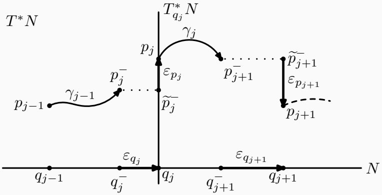


**Figure 1:** Description of $\mathcal{A}_r$ from Kragh's article

Kragh shows that these functions admit good index pairs.

# Definition of the (pre)-spectrum

Recall the relative Thom construction:

$$(A, B)^E := (\mathbf{D}(E)|_A, \mathbf{S}(E)|_A \cup (\mathbf{D}(E)|_B))$$

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Essentially, Kragh shows that if $(A_r, B_r)$ is a good index pair for $\mathcal{A}_r$, then

$$(A_{r+1}, B_{r+1}) \simeq (A_r, B_r)^{p^* TN}$$

where $p : T^*\Omega(N, \underline{r}) \to N$ is just projecting onto one of the $T^*N$ factors and then further projecting down to $N$.

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The way this is done is by looking at the diagram:

$$\begin{array}{ccc} p^* TN & \xrightarrow{\ h\ } & T^*\Omega(N, \underline{r+1}) \\ \big\downarrow & & \big\downarrow \mathcal{A}_{r+1} \\ T^*\Omega(N, \underline{r}) & \xrightarrow[\ \mathcal{A}_r\ ]{} & \mathbb{R} \end{array}$$

# Definition of the (pre)-spectrum

So far, we have defined isomorphisms

$$(A_{r+1}, B_{r+1}) \simeq (A_r, B_r)^{p^* TN}$$

We need to stabilize the bundle $p^* TN$ so that we get a suspension spectrum (recall that $\Sigma^k(A/B) = \mathrm{Th}(\mathbb{R}^k)$).

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The way this is done is by just picking a stabilization, e.g. by using a Whitney embedding $N \to \mathbb{R}^k$ which has normal bundle $\nu$. This gives maps (in fact, homotopy equivalences if $N$ is oriented) of index pairs

$$\Sigma^k(A_r, B_r)^{p^* \nu^{\oplus r+1}} = (A_r, B_r)^{p^* \nu^{\oplus r} \oplus p^* \nu \oplus p^* TN} \to (A_{r+1}, B_{r+1})^{\nu^{\oplus r+2}}$$

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If we call this $\tau_r$, this more or less defines a pre-spectrum

$$\tau_r : Z_{(r+1)k} \to Z_{(r+2)k}$$

# The spectrum

This can be modified (by taking iterating mapping cylinders and filling in the gaps) to produce an honest spectrum $Z_n$ 'representing' the symplectic cohomology of $T^*N$. In fact, one can identify

$$(A_r, B_r) \simeq (\mathbf{D}T(\Omega(N, \underline{r})), \mathbf{S}T(\Omega(N, \underline{r})))$$
$$I_r \simeq \mathrm{Th}(T(\Omega(N, \underline{r})))$$

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**Theorem (Kragh):** *The Viterbo maps can be identified with the following diagram on spectra:*

$$\begin{array}{ccc} H^\bullet(\Omega L) & \xrightarrow{\Omega j^!} & H^\bullet(\Omega N) \\ \downarrow & & \downarrow \\ H^\bullet(L) & \longrightarrow & H^\bullet(N) \end{array} \qquad \begin{array}{ccc} \Omega L^{-TL+\eta} & \xleftarrow{\Omega j_!} & \Omega N^{-TN} \\ \uparrow & & \uparrow \\ L^{-TL} & \xleftarrow[j_!]{} & N^{-TN} \end{array}$$

Thank you for your attention!

# References


- Seidel, A biased view of symplectic cohomology
- Kragh, The Viterbo transfer as a map of spectra
- Ganatra, Symplectic cohomology and duality for the wrapped Fukaya Category