Categorical Symplectic Topology of T^*S^n

Contents

0 Introduction

The aim of this essay is to study a formidable object - the so-called **Fukaya category** associated to a symplectic manifold M - along with some applications. In a sentence, the Fukaya category can be thought of as a categorification of the intersection theory of a special type of submanifold in M, the Lagrangians $L \subset M$.

Just like algebraic geometers study Ext groups of sheaves $Ext^\bullet(\mathcal{E}, \mathcal{F})$, so symplectic geometers study Floer cohomology of Lagrangians $HF^\bullet(L_0, L_1)$. This is more than just an analogy: the homological mirror symmetry conjecture states that the (derived) Fukaya category of a symplectic manifold is equivalent to the derived category of coherent sheaves on its mirror $M^\dagger$, and in fact the Floer cohomology groups appear as morphism spaces in the Donaldson-Fukaya category, whereas the Ext groups appear as morphism spaces in the latter. The original motivation for this is string theory, where the A-model studies Lagrangian submanifolds, whereas the B-model studies the complex subvarieties of a Calabi-Yau.

The remarkable thing about Fukaya categories is that they can be calculated. In particular, Abouzaid has shown that the wrapped Fukaya category of a cotangent bundle is generated by a cotangent fibre. In this essay, we will restrict ourselves to a modest version of this result, concerning the (exact) Lagrangian submanifolds of T^*S^n.

The outline of the essay is as follows: we begin with a lightning review of the basic concepts of symplectic geometry, and state the important Weinstein neighbourhood theorem, which will be incredibly useful later on in the definition of the Dehn twist. We also calculate the fundamental group of the Lagrangian Grassmanian, which will be put to use in the definition of the Maslov index and the grading of the Floer cohomology groups.

In chapter 2, we give a tour of Lagrangian Floer cohomology, which also attempts to be historically accurate. We begin with an overview of Morse's original vision of Morse theory, which studies the topology of a manifold using the critical points of a smooth function on it. This strategy completely determines the homotopy type of the manifold, which can be described by attaching cells every time a critical value is passed. However, a more modern approach due to Smale identifies the presence of **moduli spaces** of gradient trajectories between critical points. Counting such gradient trajectories between critical points turns out to be exactly analogous to the differential in celullar homology, and hence we obtain the result that Morse homology recovers the Betti homology. It was these ideas of Morse and Smale, as well as Witten, which led Andreas Floer to his infinite dimensional Morse theory. In a nutshell, one can study the paths γ between two Lagrangian submanifolds (in physics terminology, these are strings between D-branes). This is an infinite-dimensional path space, and comes equipped with an *action functional*, which measures the symplectic area sweeped out by an infinitesimal deformation of γ. The ingenious idea of Floer was to apply Morse theory in this context - the critical points turn out to be constant paths at intersection points, and the gradient trajectories become **J-holomorphic strips**. Trying to generalize Morse theory to this inifinite dimensional context comes with considerable analytic difficulties (which we will mostly omit). However, the main reason the analysis works out is that the equations defining J-holomorphic curves are elliptic PDE's, whose linearizations are Fredholm operators, to which an infinite-dimenisonal inverse function theorem applies.

With all of this in mind, chapter 2 ends with a definition of Floer cohomology and subsequently the Fukaya category, which has objects L (the Lagrangian submanifolds) and morphism spaces $\mathrm{Hom}(L_0, L_1) := CF(L_0, L_1)$. The name of the Fukaya category is deceiving, as it is resolutely not a category in the usual sense: its morphism spaces are chain complexes equipped with a differential μ^1, and there are composition maps μ^2 which are associative only up to a homotopy given by a higher morphism μ^3. This process continues to infinity, which makes the Fukaya category an A_∞ category. All of these higher operations are defined by counting J-holomorphic strips with $k+1$ marked points, and enjoy remarkable recursive combinatorial properties, due to the fact that the domain spaces (disks with marked point) form operads, called *Stasheff associahedra*.

In chapter 3, we embark on trying to prove the long exact sequence in Floer cohomology due to Seidel, which states that for an exact Lagrangian sphere $Z \subset M$ and any Lagrangians A, B, the following is exact:

$$... \to HF^\bullet(Z,A) \otimes HF^\bullet(Z,B) \to HF^\bullet(\tau_Z(A), B) \to HF^\bullet(A,B) \to ...$$

τ_Z denotes a symplectic automorphism known as a **Dehn twist**. In order to prove this result, we need to go through a little detour in Picard-Lefschetz theory, which is what happens when one tries to extend Morse theory to the holomorphic setting. Here, passing through critical values is replaced by the notion of monodromy around the critical value. Importantly, the Dehn twist along a vanishing cycle appears as the monodromy around a single critical point of a Lefschetz fibration! Conversely, any Dehn twist appears as the monodromy of a certain standard fibration. Hence, if we want to study Dehn twists, we might as well study these fibrations.

With this motivation, we come to Seidel's TQFT. Namely, instead of just counting J-holomorphic maps from a surface to a symplectic manifold, we can count J-holomorphic sections of Lefschetz fibrations $E \to S$ over surfaces, which have certain Lagrangian boundary conditions. When the base is a surface with strip-like ends, labelled with positive and negative edges, this will result in **relative Gromov-Witten invariants**, which count sections asymptotic to y_e at the ends of the strip:

$$C\Phi_{E/S} : \bigotimes CF(L_{e^+,0}, L_{e^+,1}) \to \bigotimes CF(L_{e^-,0}, L_{e^-,1})$$
$$\otimes y_{e^+} \mapsto \sum \Phi_{E/S}(y_{e^-}, y_{e^+}) \otimes y_{e^+}$$

As a particular application, the trivial fibration will recover the Floer cohomology groups. Moreover, the pair-of-pants surface will recover the multiplication μ^2. Using this, as well as the standard fibration for the Dehn twist, the relative invariants allow for the construction of chain maps which lead to Seidel's LES in Floer cohomology.

Finally, in chapter 4, we use all of the machinery developed to prove that every exact Lagrangian inside T^*S^n behaves Floer-theoretically like the zero section, in the sense that they are isomorphic objects in the (derived) Fukaya category. Informally, an isomorphism in the Fukaya category means that Floer theory cannot distinguish between the two objects. This is a weak version of the nearby Lagrangian conjecture, which states that all exact Lagrangians (satisfying some extra conditions) are Hamiltonian isotopic to the zero section. The main ingredient in the course of the proof will be the long exact sequence in Floer cohomology, along with a result from the representation theory of quivers.

1 Symplectic geometry

1.1 Basics

We introduce the basic objects of symplectic geometry, together with a few theorems.

Definition 1.1 (Symplectic manifold): *A symplectic manifold (M, ω) is a manifold equipped with a nondegenerate, closed 2-form ω.*

The closedness is a fundamental assumption, which shows that the symplectic area is homotopy invariant (the importance of this will become apparent later on)

Example : Even-dimensional Euclidean space with coordinates (x_i, y_i) comes equipped with the two form

$$\omega = \sum_i dx_i \wedge dy_i$$

In fact, this can be generalized to all cotangent bundles T^*M, where the local coordinates on M are given by q_i and the fibre coordinates by p_i. Then we have an exact symplectic form given locally as follows:

$$\omega := \sum dp_i \wedge dq_i = d\lambda,$$
$$\lambda = \sum p_i dq_i$$

By linear algebra, any symplectic manifold has even dimension. Moreover, the symplectic form, just like the Riemannian metric, gives an isomorphism between the tangent and cotangent bundles and we can define a Hamiltonian vector field as a vector field dual to an exact 1-form:

$$\omega(X_H, -) = dH$$

The flow of this is a **Hamiltonian isotopy**, which preserves the symplectic form due to the magic formula:

$$\mathcal{L}_{X_H}\omega = d\iota_{X_H}\omega + \iota_{X_H}d\omega = d^2H = 0$$

So $\frac{d}{dt}\phi_t^*\omega = \phi_t^*\mathcal{L}_{X_H}\omega = 0$

In the course of the next chapter, we will develop a cohomology theory which is invariant under these special types of symplectomorphisms. This theory, called Floer cohomology, is extremely important in the study of a special class of submanifolds of symplectic manifolds, the so-called Lagrangians.

Definition 1.2 (Lagrangian submanifold): *A submanifold $L \subset M$ is called **Lagrangian** if it has half the dimension of M and the inclusion pulls back the symplectic form to 0: $\omega|_L = 0$. Equivalently, at every point $p \in L$, T_pL is a Lagrangian subspace of T_pM, which can also be used to show that $\mathcal{N}_{L/M} \simeq T^*L$*

Example : An important example of a Lagrangian embedding is the zero section $s_0 : M \to T^*M$.

Example : There is no Lagrangian 2-sphere inside $\mathbb{CP}^2$, since if there was such an object, its normal and tangent bundles would be isomorphic. But the Euler class of the tangent bundle, evaluated against the fundamental class of S^2, would give (by Gauss-Bonnet) $\chi(S^2) = 2$, whereas the Euler class of the normal bundle is Poincare dual to S^2 and hence its Euler number, being the intersection number of S^2 with itself, would end up being a square integer!

Definition 1.3 (Lagrangian Grassmanian): *We denote the set of linear Lagrangian subspaces of $\mathbb{R}^{2n}$ by $\mathcal{L}(n)$.*

In fact, the set of unitary matrices $U(n)$ acts transitively on $\mathcal{L}(n)$, since every Lagrangian can be mapped via a unitary matrix to the horizontal Lagrangian $\Lambda_{hor} := \{y = 0\}$. The stabilizer of this is precisely $O(n)$, since a matrix $X + iY = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$ sends Λ_{hor} into itself precisely when $Y = 0$ i.e. the matrix is in $O(n)$. This shows that $\mathcal{L}(n) \simeq U(n)/O(n)$.

Proposition 1.4 (Fundamental group of Lagrangian Grassmanian): *The square of the determinant map induces an isomorphism on fundamental groups*

$$\pi_1(\mathcal{L}(n)) \simeq \pi_1(U(n)/O(n)) \xrightarrow{\det^2} \pi_1(S^1) \simeq \mathbb{Z}$$

Proof. We have the following fibrations:

$$\begin{array}{ccc} SU(n-1) & \longrightarrow & SU(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array} \qquad \begin{array}{ccc} SO(n) & \longrightarrow & SU(n) \\ & & \downarrow \\ & & SU(n)/SO(n) \end{array} \qquad \begin{array}{ccc} SU(n)/SO(n) & \longrightarrow & U(n)/O(n) \\ & & \downarrow \det^2 \\ & & S^1 \end{array}$$

The vertical arrow in the first one sends a matrix to its first column. This fibration, using the long exact sequence of homotopy groups, tells us by induction that $SU(n)$ is simply-connected. Applying the same LES to the second one, we get that $\pi_1(SU(n)/SO(n)) = 0$ is also simply-connected. Finally, applying the LES to the third fibration gives us the desired isomorphism. $\square$

Hence, given any loop of linear Lagrangian subspaces, we can associate its **Maslov index**, which is the integer in the isomorphism above. Note that since $H_1 = \pi_1^{ab}$,

$$H^1(\mathcal{L}(n);\mathbb{Z}) \simeq \mathrm{Hom}(H_1(\mathcal{L}(n)),\mathbb{Z}) = \mathrm{Hom}(\pi_1(\mathcal{L}(n)),\mathbb{Z})$$

Pulling back the generator of $\pi_1(S^1)$ along $\det^2$ we get $\mu \in H^1(\mathcal{L}(n);\mathbb{Z})$, the universal Maslov class.

Let's also quickly mention the fact that $Sp(2n)$, the group of matrices preserving the standard symplectic form, deformation retracts onto $U(n)$, and hence the classifying spaces of symplectic and complex vector bundles are homotopy equivalent:

$$BSp \sim BU \implies H^\bullet(Sp) \simeq H^\bullet(U)$$

Therefore symplectic manifolds have the same characteristic classes as complex manifolds. We can thus talk about Chern classes in the symplectic setting.

1.2 Symplectic invariants

The geometry of symplectic manifolds is both rigid and flexible at the same time. This flexibility is illustrated by the following theorems, whose proofs can be found in [6], chapters 8 and 9.

Theorem 1.5 (Darboux): *Let (M,ω) be a symplectic form and $p \in M$ be any point. Then there is a local coordinate system in which ω looks like the standard form on Euclidean space.*

Theorem 1.6 (Weinstein tubular neighbourhood): *Given a Lagrangian $L \subset (M,\omega)$, there exists a neighbourhood U of L in M and U' of L' (the zero section) in T^*L and a symplectomorphism $\varphi : U \simeq U'$ such that*

$$\begin{array}{ccc} U & \xrightarrow{\ \varphi\ } & U' \\ & {\scriptstyle\iota}\nwarrow \quad \nearrow{\scriptstyle s_0} & \\ & L & \end{array}$$

These theorems basically tell us that symplectic manifolds do not admit local invariants near points and near Lagrangian submanifolds.

The essential ingredient in studying invariants of symplectic manifolds came from Gromov's insight of counting maps from Riemann surfaces. The point is that every symplectic manifold can be made into an almost complex manifold compatibly with the symplectic form: there exists a contractible (in fact convex) set of endomorphisms $J : TM \to TM$ such that $g := \omega(-,J-)$ is a Riemannian metric on M. (One could also get away with studying only the tame almost complex structures). Importantly, we can move around in this space of compatible almost complex structures as to end up in a generic situation (transversality), which will help in the definition of Floer cohomology.

Definition 1.7 (J-holomorphic map): *A map $f : (\Sigma, j) \to (M,J)$ is J-holomorphic if it commutes with the almost complex structures, i.e. $df \circ j = J \circ df$. Applying J to both sides this is equivalent to $df + J \circ df \circ j = 0$, i.e. f is a zero of the generalized del bar operator*

$$\overline{\partial}_J := \frac{1}{2}(d + J \circ d \circ j)$$

The J-holomorphicity equation can be thought of as a generalized Cauchy-Riemann equation and in the next chapter we will study the moduli space of its solutions. Since this operator is elliptic, its linearization is Fredholm and hence the expected dimension of the moduli space, by an infinite dimensional version of the inverse function theorem, will be the index of the operator. [1] It turns out that this index, à la Atiyah-Singer, is a purely topological invariant, defined using the Maslov index, which we mention in the next section.

1.3 The Maslov index of a bundle pair

Let Σ be an oriented compact surface with boundary $\partial\Sigma = \bigcup_1^h \partial_k\Sigma$ consisting of a finite number of copies of S^1 - the examples we will mostly be using are the unit disk D and the cylinder $C = S^1 \times I$. Any symplectic vector bundle $E \to \Sigma$ is symplectically trivial, if we assume that the boundary is nonempty (see Proposition 2.6.7 in [15]). If furthermore $\lambda \to \partial\Sigma$ is a Lagrangian subbundle, in the sense that the fibers λ_p are Lagrangians inside E_p for $p \in \partial\Sigma$, then a trivialization $\Phi : E \simeq \Sigma \times \mathbb{R}^{2n}$ restricted to the boundary components $\partial_1\Sigma, ..., \partial_h\Sigma$ produce loops $\Phi(\lambda|_{\partial_i\Sigma})$ which we denote $\gamma^i_{\Phi,\lambda} : S^1 \to \mathcal{L}(n)$.

Definition 1.8 (Maslov index of a symplectic bundle pair): *We denote by*

$$\mu(E,\lambda) := \sum_i \mu(\gamma^i_{\Phi,\lambda})$$

the sum of the Maslov indices of the boundary components. This value is independent of the trivialization Φ and is called the Maslov index of the symplectic bundle pair (E,λ).

The reason for the independence of Φ is that two trivializations Φ_1, Φ_2 differ by some map $g : \Sigma \to Sp(2n)$ and hence $\mu(\Phi_2, \partial_i\Sigma) = \mu(\Phi_2, \partial_i\Sigma) + 2\mathrm{ind}(g|_{\partial_i\Sigma})$. Summing over all i, the indices of g are going to add up to 0, since the surface Σ provides a nullcobordism between the boundary components and the degree map is invariant under cobordisms (see Proposition 2.1.14 in [10]).

The situation we are most interested in is when we have a map $f : (\Sigma, \partial\Sigma) \to (M, L)$, where M is a symplectic manifold and L is a Lagrangian. This then this gives us a symplectic bundle pair $(f^*TM, f|^*_{\partial\Sigma}TL)$ whose Maslov index we denote by $\mu_L(f)$. In particular, when $\Sigma = D$, we get:

Definition 1.9 (Maslov class): *Given $u \in \pi_2(M,L)$ represented by a map $u : (D, \partial D) \to (M, L)$, its restriction to the boundary is a loop γ in L. This loop $\gamma : S^1 \to L$ defines a loop of Lagrangians $T_{\gamma(t)}L$ inside the Lagrangian bundle $\mathcal{L} \to M$, whose fiber consists of the Lagrangians inside T_pM. Since $u^*\mathcal{L}$ is trivial, the unit disk being contractible, we can use a trivialization to identify each fiber with $\mathcal{L}(n)$ and we get a loop in $\mathcal{L}(n) \simeq \mathbb{Z}$. This defines the Maslov class*

$$\mu_L : \pi_2(M, L) \to \mathbb{Z}$$

[1] The proper formulation of this needs the notion of a section of a Banach bundle, but we will not be concerned with the details of the analysis.

2 Floer theory

We will give a brief overview of Morse theory, starting from the historically first point of view, namely that of Morse, which is still powerful enough to express the homotopy type of spaces just by looking at critical points of smooth functions. Then, we explain a slightly more modern approach, using moduli spaces of flowlines, which was discovered by Smale. This will lead us to the concept of Morse homology, and more generally to an infinite-dimensional version due to Floer, which is conceptually similar, but whose analytical foundations are quite a bit more difficult. This will allow us to define Lagrangian Floer cohomology, which occur as morphism spaces in the Donaldson-Fukaya category.

2.1 Morse theory and Morse homology

2.1.1 Morse theory

The basic idea of Morse theory is to study the topology of a smooth manifold by exhibiting the critical points of a generic function on it. For example, for any manifold nicely embedded in $\mathbb{R}^N$, one can take the height function, and every time one hits a critical point, the topology changes by adding a cell with dimension equal to the index of the Hessian (more on this in a moment).

More precisely, start with a smooth function $f : M \to \mathbb{R}$.

Definition 2.1 (Hessians and critical points):

- *A critical point of f is a point $p \in M$ where $df_p = 0$.*
- *The Hessian of f at a critical point p in local coordinates x_i is given by the bilinear form on T_pM:*
$$\mathrm{Hess}_p(f)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$
More globally, one can define it as $\mathrm{Hess}_p(f)(X,Y) = (X \cdot Y \cdot f)(p) = (Y \cdot X \cdot f)(p)$. The last equality follows, as their difference is $[X,Y] \cdot f = df_p([X,Y]) = 0$, as p is a critical point.
- *p is a non-degenerate critical point if $Hess_p(f)$ is a nondegenerate bilinear form.*

The idea is that degenerate critical points kill off a bit of the tangent space and lose geometric information. In contrast, the Morse lemma (see e.g. [16]) tells us that around a neighbourhood of a non-degenerate critical point p the function f looks like

$$f = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{n} x_j^2$$

In particular, this shows that the non-degenerate critical points are isolated.

The index of f at p is the number k, which can be thought of as the dimension of the maximal subspace where the Hessian is negative definite, i.e. the number of negative eigenvalues of the Hessian. We will illustrate the meaning of this in the following example:

Example (CW structure on complex projective space): Let us identify $\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times \simeq S^{2n+1}/S^1$. This has coordinates $[z_0 : ... : z_n]$ such that $\sum |z_j|^2 = 1$ and the brackets mean that we can rescale by any $\lambda \in S^1$. Define

$$f : \mathbb{CP}^n \to \mathbb{R}$$
$$[z_0 : ... : z_n] \mapsto \sum j|z_j|^2$$

The coefficient j here doesn't have much significance - we can pick any $n+1$ real, distinct constants just as well. On the affine open sets U_i where $z_i \neq 0$, one has a standard coordinate system

$$U_i \simeq \mathbb{C}^n$$
$$[z_0 : ... : z_n] \mapsto |z_i|\frac{z_j}{z_i} = x_j + iy_j$$

Note that we have to multiply by $|z_i|$ as we normalized the coordinates to be in S^{2n+1}. Then $|z_j|^2 = x_j^2 + y_j^2$ for $i \neq j$ and $|z_i|^2 = 1 - \sum(x_j^2 + y_j^2)$. This implies that f can be written as

$$f = i + \sum_{j \neq i} (j-i)(x_j^2 + y_j^2)$$

in the neighbourhood U_i. The only critical point in this neighbourhood is then $[0 : .. : 1 : ... : 0]$, where the 1 occurs only in the i-th coordinate, and has index twice the number of negative integers among $\{j - i | j = 0, ..., n\}$. Hence, the index is $2i$, and we thus recover the fact that $\mathbb{CP}^n$ has a cell decomposition with one cell in each dimension $0, 2, 4, ..., 2n$.

The last line of the example has to do with the following fact: in between two critical values $a, b \in \mathbb{R}$ the topology of $f^{-1}(a,b) \subset M$ doesn't change, as the gradient flow produces an isotopy $\psi_{b-a} : f^{-1}(-\infty, b) \simeq f^{-1}(-\infty, a)$. However, once we pass through a critical point of index k, a cell of dimension k is attached.

Another way to think about this is that the index gives the dimension of the descending manifold at p, which consists of the set of points which emanate from p down using the negative gradient flow. Similarly, the ascending manifold at p is the set of points which flow down to p using the gradient flow. These ideas lead to the concept of a moduli space of flow lines between two critical points, which we discuss now.

2.1.2 Morse homology

Let us now be a bit more precise about the remarks at the end of the previous section. Equip M with a Riemannian metric g. The gradient vector field ∇f is dual, via g, to the one-form df:

$$g(\nabla f, -) = df$$

Denote its flow be ψ_t. The set of critical points of f, assuming M is compact, is finite and we can consider the moduli spaces of flow lines between critical points.

$$\hat{\mathcal{M}}(p,q) = \{x \in M | \lim_{t\to\infty} \psi_t(x) = q, \lim_{t\to-\infty} \psi_t(x) = p\}$$

We would like to say that this is the intersection $\mathcal{D}(p) \cap \mathcal{A}(q)$ of the descending manifold at p and the ascending manifold at q. However, we run into a problem: these two manifolds may not be transverse. Consider the following picture:

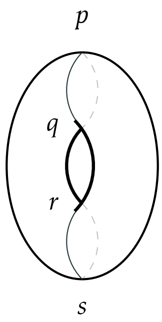


Figure 1: Failure of Morse-Smale condition

This depicts the torus with the height function, which is Morse. However, $\mathcal{D}(q)$ is the same as $\mathcal{A}(r)$, so they are not transverse! If one perturbs the function a little bit though, then $\mathcal{D}(q)$ will connect q to s and $\mathcal{A}(r)$ will connect p to r, resulting in disjoint, hence transverse, cycles. So while the height function was Morse and gave us the information about the homotopy type of the torus, it doesn't allow us to talk about moduli of flow lines.

To fix this, we impose a generic condition, called the Morse-Smale condition, on f. It implies that these manifolds are transverse and is equivalent to df being transverse to the 0-section in T^*M. We can now define

$$\mathcal{M}(p,q) := \hat{\mathcal{M}}(p,q)/\mathbb{R}$$

to be the space of flow lines between p and q, modulo reparametrization. If f is Morse-Smale, this becomes a smooth manifold of dimension $\text{ind}(p) - \text{ind}(q) - 1$.

We can use this to define a chain complex as follows:

Definition 2.2 (Morse-Smale-Witten complex):

$$C_k = \bigoplus_{p \in Crit(f)} \mathbb{Z} \cdot p \textit{ with differential } dp = \sum_{ind(p,q)=1} \#\mathcal{M}(p,q)q$$

Here, we define the relative index as $\text{ind}(p,q) := \text{ind}(p) - \text{ind}(q)$. We can either use $\mathbb{Z}/2$ coefficients, or have to count the flow lines with a given orientation. The reason that the differential squares to zero is that the moduli space of flow lines can be compactified to a 1-dimensional manifold, whose oriented boundary is empty!

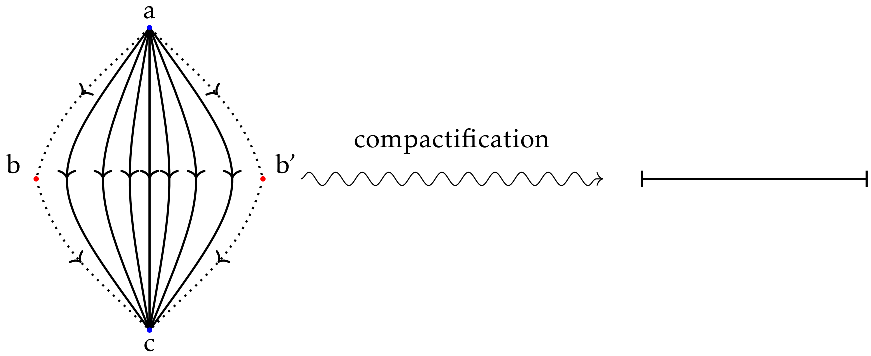


Figure 2: broken flow lines

More precisely, the coefficient of c such that $\text{ind}(a,c) = 2$ in $d^2 a$ is given by

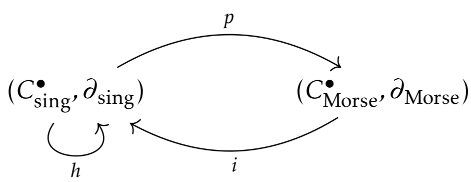
$$\sum_{\text{ind}(a,b)=\text{ind}(b,c)=1} \#\mathcal{M}(a,b)\#\mathcal{M}(b,c)$$

This vanishes, as the moduli space of flow lines between a and c is an open 1-manifold (since $\text{ind}(a,b)-1 = 1$) whose compactification has boundary precisely the broken flow lines as in figure 2 (one needs some analysis to prove this last "gluing" statement). Importantly, one has to use the fact that any flow line starts and ends at a critical point. Later on, when we deal with Lagrangian Floer cohomology, this will not be true, and we will need an extra assumption (finite energy).

We can see that the cellular complex and Morse complex, for the cell structure on M induced by f, agree as graded vector spaces, as they are both free on the number of critical points of a given index. The cells are given by compactifications of descending manifolds and in fact counting flow lines between critical points whose indices differ by one is akin to the boundary map in cellular homology! Hence Morse homology is the same as Betti homology (see Theorem 7.4 in [5] or Appendix 4.9 in [3]):

$$MH_\bullet(M) \simeq H_\bullet(M)$$

Remark: One may just as well define a Morse cohomology, by reversing the direction of the flowlines. Then, we still have $MH^\bullet \simeq H^\bullet$. Moreover on the chain level, the Morse cochains form a deformation retract of the singular cochains: there are maps



such that $\text{id} - ip = \partial_{\text{sing}} h + h\partial_{\text{sing}}$. The singular cochains have a cup product, making them an associative dg algebra. When one tries to carry this over to the Morse cochains, the product becomes associative only up to homotopy, endowing the Morse cochains with an A_∞-algebra structure. For more on this, see [11] and [14].

2.2 Lagrangian Floer cohomology

We will consider an infinite generalization of Morse theory, in the sense that the function is defined on an infinite dimensional manifold (a path space). In a single sentence, Lagrangian Floer homology is Morse theory for the action functional which measures the symplectic area of a J-holomorphic strip.

2.2.1 The action functional and monotonicity

Let L_0, L_1 be two compact Lagrangians intersecting transversely inside a symplectic manifold (M, ω). We consider the space of all paths starting at L_0 and ending at L_1. In fact, we will have to consider a certain subcover of its universal cover, an element of which is given by $\gamma \in \mathcal{P}(L_0, L_1)$ together with a homotopy class of a path Γ from a fixed basepoint γ_0 to γ, i.e. $\Gamma : [0,1] \times [0,1] \to M$ subject to $\Gamma(0,-) = \gamma_0, \Gamma(1,-) = \gamma$. We denote the connected component of γ_0 by $\mathcal{P}(L_0, L_1; \gamma_0)$. Hence, elements of the universal cover will be pairs (γ, Γ), but we will have to mod out by a certain equivalence relation, which we now describe.

We would like to measure the symplectic area between the two paths

$$\mathcal{A} : \tilde{\mathcal{P}}(L_0, L_1) \to \mathbb{R}$$
$$\mathcal{A}(\gamma, \Gamma) = \int_\Gamma \omega = \int_{[0,1]\times[0,1]} \Gamma^* \omega$$

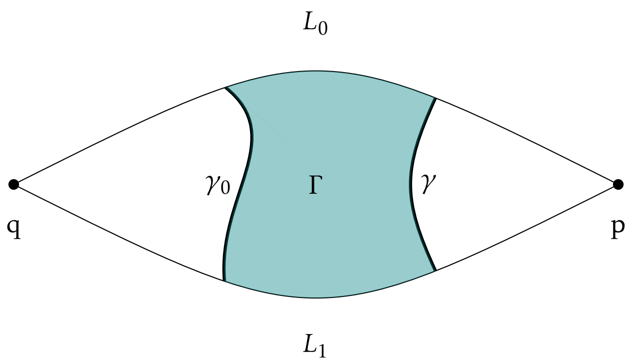


Figure 3

However, a priori, this is not well defined. If (γ, Γ') is another pair, we see that

$$\mathcal{A}(\gamma, \Gamma) - \mathcal{A}(\gamma, \Gamma') = \int_C H^* \omega$$

where $\overline{\Gamma} \# \Gamma' = H : C \to M$ is a map from the cylinder obtained by gluing the boundaries of the two squares on which Γ and Γ' agree. We can also think of this as a loop in the path space, starting and ending at γ_0. By the closedness of ω, this is independent of the homotopy class of H and defines an integration homomorphism

$$I_\omega : \pi_1(\mathcal{P}(L_0, L_1; \gamma_0)) \to \mathbb{R}$$
$$I_\omega(H) = \int_H \omega$$

Furthermore, we also have that $(H^*TM, h_0^*TL_0 \sqcup h_1^*TL_1)$ is a symplectic bundle pair, where h_i are the restrictions of H to the top and bottom of the cylinder. This then has a Maslov index 1.8:

$$I_{\mu,L_0,L_1} : \pi_1(\mathcal{P}(L_0, L_1; \gamma_0)) \to \mathbb{Z}$$
$$I_{\mu,L_0,L_1}(H) = \mu(H^*TM, h_0^*TL_0 \sqcup h_1^*TL_1)$$

Now might be a good time to introduce the concept of a monotone triple:

Definition 2.3 (Monotone triples): *We say (M, L_0, L_1) is monotone if the two homomorphisms I_ω and I_{μ,L_0,L_1} defined above are proportional by a real number λ. We denote by N_μ the minimal Maslov number, i.e. the positive generator of $\operatorname{im} I_\mu$. Similarly, we define Σ_i to be the positive generators of I_μ restricted to $\pi_2(M, L_i)$.*

Remark (Monotone Lagrangian submanifolds): There is an analogous notion of a monotone Lagrangian submanifold, where the cylinder gets replaced by a disk, so we have only one boundary component and a map $f : (D, \partial D) \to (M, L)$. The Maslov homomorphism is then defined via the Maslov index of the pair $(f^*TM, f|_{\partial D}^* TL)$.

The discussion above tells us that in order to define the action functional, we need to mod out by the relation $(\gamma, \Gamma) \sim (\gamma, \Gamma')$ whenever $I_\omega(\overline{\Gamma} \# \Gamma') = 0$. Furthermore, we mod out by $(\gamma, \Gamma) \sim (\gamma, \Gamma')$ whenever $I_\mu(\overline{\Gamma} \# \Gamma') = 0$, to ensure the grading works out.

Definition 2.4 (Novikov cover): *The resulting cover, the Novikov covering space, is denoted $\tilde{\mathcal{P}}(L_0, L_1; \gamma_0)$*

2.2.2 Comparison with Morse theory

The action functional is now well-defined on the Novikov cover. In analogy with Morse theory, we would like to study the critical points of this action functional (the points where $d\mathcal{A} = 0$), as well as its "gradient flow lines". These will turn out to be the constant paths at intersection points, together with J-holomorphic strips connecting intersection points.

More precisely, there is a closed action 1-form defined by

$$T_\gamma \mathcal{P}(L_0, L_1; \gamma_0) \to \mathbb{R}$$
$$\alpha_\gamma(\xi) = \int_0^1 \omega(\dot{\gamma}(t), \xi(t)) dt,$$

Note that the covering map π is a local diffeomorphism and hence we can identify the tangent spaces of $\tilde{\mathcal{P}}(L_0, L_1; \gamma_0)$ and $\mathcal{P}(L_0, L_1; \gamma_0)$. Moreover, a tangent vector ξ on the path space is a vector field along the path γ. This is related to the action functional as follows:

$$d\mathcal{A} = -\pi^*\alpha$$

where $\pi : \tilde{\mathcal{P}}(L_0, L_1) \to \mathcal{P}(L_0, L_1)$ is the projection map[2].

This shows that the critical points of the action functional are precisely the constant loops at $p \in L_0 \cap L_1$. If we want to create a Morse-type complex, we now need to understand the gradient flows.

Choosing a generic family of compatible almost complex structures J_t on M allows us to define an L^2 metric g on $T_\gamma \mathcal{P}(L_0, L_1)$ by integrating the associated Riemannian metric over all t:

$$\langle \zeta, \xi \rangle = \int_0^1 \omega(\zeta(t), J_t \xi(t))\, dt$$

Let's consider the negative gradient flow of $\mathcal{A}$. This is a vector field on $\mathcal{P}(L_0, L_1)$ and its value at γ is a tangent vector in $T_\gamma \mathcal{P}(L_0, L_1)$ i.e. a vector field along γ. By the definition of our metric, we must have that

$$g(\nabla \mathcal{A}, -) = d\mathcal{A}$$

In other words, by using the formula for $d\mathcal{A}$ we must have that for all ξ:

$$\int_0^1 \omega(\nabla \mathcal{A}_\gamma(t), J_t \xi(t)) dt = \int_0^1 \omega(\dot\gamma(t), \xi(t)) dt$$

Hence, by compatibility of ω with J_t and nondegeneracy, we get that $\nabla \mathcal{A}_\gamma(t) = J_t(\gamma(t))\dot\gamma(t)$.

Now, a gradient trajectory u is a curve in the path space interpolating between constant paths, i.e. it is a map $u : \mathbb{R} \to \mathcal{P}(L_0, L_1)$. With this in mind, if the coordinate on $\mathbb{R}$ is denoted by s the negative gradient flow equation becomes

$$\dot u = \partial_s u = -\nabla \mathcal{A}_u = -J_t(u)\partial_t u$$

All in all, we can package this information as follows:

- J-holomorphicity: $\partial_s u + J_t \partial_t u = 0$
- Lagrangian boundary condition: $u(s, 0) \in L_0$, $u(s, 1) \in L_1$
- Asymptotic behaviour: $\lim_{s \to \pm\infty} u = p, q$

[2] This is true because of the following calculation: firstly, the differential of the action functional is $d\mathcal{A}_\gamma(\xi) = \frac{\partial}{\partial s}\mathcal{A}(\gamma, \Gamma)|_{s=0}$ where $\xi = \partial_s \Gamma = d\Gamma(\partial_s)$ is a vector field along γ. Differentiating in the integral and using Stokes' we get:

$$\int_{I^2} (\partial_s \Gamma^* \omega)|_{s=0} = \int_{I^2} \Gamma^*(\mathcal{L}_\xi \omega) = \int_{I^2} \Gamma^*(d\iota_\xi \omega) =$$
$$= \int_I \gamma^*(\iota_\xi \omega) = -\int_0^1 \omega(\dot\gamma(t), \xi(t))\, dt$$

- We also impose the finite energy condition: $E(u) = \int u^*\omega < \infty$ (as mentioned before, this is needed to ensure that every gradient trajectory starts and ends at a critical point)

Note that, by the asymptotic condition, we can extend u to a map $(D^2, \partial D^2) \to (M, L_0 \cup L_1)$ and hence represents an element of $\pi_2(M, L_0 \cup L_1)$.

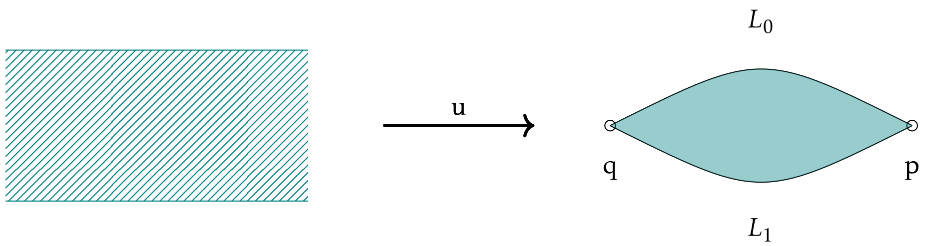


Figure 4: A 'gradient trajectory' between two 'critical points' is a J-holomorphic strip

2.2.3 Moduli spaces of strips and the Floer cochain complex

We now need to understand the moduli of gradient trajectories, which in our new language means moduli of J-holomorphic strips between p and q. Unfortunately, things are not as simple as in the case of Morse homology - if we are not careful in the way we count these strips and naively define $dp = \sum_{ind(p,q)=1} \#\mathcal{M}(p,q)\, q$ we immediately run into some problems:

- Why is the sum well-defined?
- What is the index and the grading?
- Moreover, why is it even a chain complex?

Well-definedness: Indeed, the sum is not well-defined and may be infinite. To make sure this doesn't happen, we count pseudoholomorphic strips representing a fixed class $[u] \in \pi_2(M, L_0 \cup L_1)$ (all of these have the same symplectic area due to Stokes' theorem), and we denote the space of these solutions (up to reparametrization, i.e. modding out by $\mathbb{R}$) by $\mathcal{M}(p,q;[u])$. Then, to keep track of the symplectic area of any such class, we insert a formal variable $T^{\omega \cdot [u]}$. By Gromov compactness, there are only finitely many homotopy classes of J-holomorphic strips below a given symplectic area, so this will lead to the sum being well-defined with coefficients in the Novikov field Λ, the field consisting of formal sums $\sum a_i T^{\lambda_i}, \lambda_i \to \infty$.

Index and grading: Now, to take care of the second problem, recall that the index in Morse theory is measuring the dimension of the moduli space of gradient trajectories, before modding out by $\mathbb{R}$. In our case, putting transversality issues aside, the dimension of this moduli space is given by the index of the linearization of $\overline{\partial}_J$, whch is a Fredholm operator. This index can also be computed using the Maslov index, denoted by $\text{ind}([u])$, which is a topological invariant. The fact that the index of an elliptic operator is a topological invariant is highly nontrivial, and comes from Atiyah-Singer's index theory.

The idea in the definition of $\text{ind}([u])$ is similar to the Maslov homomorphism I_{μ,L_0,L_1}. However, thinking of u as a map from the surface with strip-like ends $[0,1]\times\mathbb{R}$, we cannot appeal to the Maslov index of a symplectic bundle pair 1.8, as this is not a compact surface. To remedy this, we need to compose with canonical short paths on the ends of the strips, where u asymptotes to p and q.

In practical terms, we can trace around the ends of the strip $\mathbb{R}\times[0,1]$ and produce a loop of Lagrangians as follows: go from T_pL_0 to T_qL_0, then follow up by a canonical short path to T_qL_1, then do the same in reverse for L_1, i.e. go to T_pL_1 and finally return to T_pL_0. The resulting element of $\pi_1(\mathcal{L}(n))\simeq\mathbb{Z}$ is the index of $[u]$.

Example (Index 1 strip): In the figure below, we start with a purple vector in the tangent space T_qL_1 then move to T_pL_1, which is then moved via the canonical short path (multiplication by i) to the green vector in T_pL_0 and to T_qL_0 and finally rotate again to end up in the black vector. This can be thought of as a full rotation of the horizontal Lagrangian $\mathbb{R}=\{y=0\}$ inside $\mathbb{R}^2$. Another way to think about the index is to count the number of times the Lagrangian tangent spaces are non-transverse - in the picture on the right, this occurs only once, precisely at the midpoint.

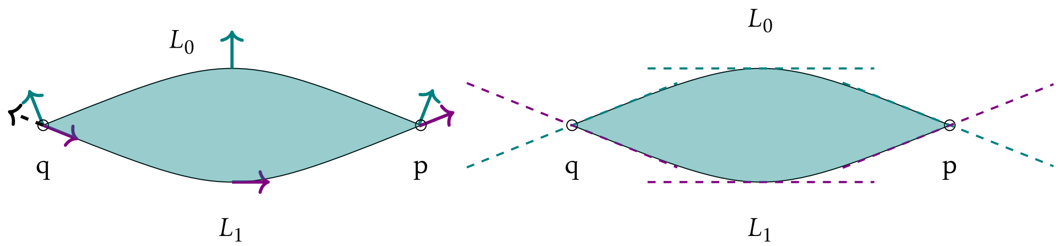


Figure 5: A strip with Maslov index 1

Now, for the matter of grading, we would like to assign integers $\deg(p),\deg(q)$ such that $\text{ind}[u]=\deg(p)-\deg(q)$. However, let us see what can go wrong.

Given a J-holomorphic strip u between p and q, and an element $v:(D,\partial D)\to(M,L_0)$ or a sphere $w:S^2\to M$ we can form the connected sums $u\#v, u\#w$, as in figure 6. In particular, we can see how the index has changed (this is Theorem 3 in Floer's paper [7]):

$$\text{ind}([u\#v])=\text{ind}([u])+\mu_{L_0}(v)$$
$$\text{ind}([u\#w])=\text{ind}([u])+2\langle c_1(TM),w_*[S^2]\rangle$$

The first equality is fairly straightforward, whereas to see the second one, one has to first note that the quotient map $U(n)\to U(n)/O(n)\simeq\mathcal{L}(n)$ induces multiplication by 2 on fundamental groups, as the first group has det inducing an isomorphism on π_1, whereas the second one has

$\det^2$ (recalling 1.4). Then, the equality follows by the fact that c_1 is measuring the difference in the trivializations between $(u\#w)^*TM$ and u^*TM on the boundary of the strip $I \times \mathbb{R}$, which are being used to identify the tangent spaces as Lagrangians inside $\mathcal{L}(n)$ and calculate the Maslov index. In other words, the first Chern class is measuring the degree of the change of trivialization map $S^1 \to Sp(2n)$ from the circle along which the sphere w and strip v are glued - for this interpretation of the first Chern class, see [15], chapter 2.7.

Since both of these quantities are meant to be equal to $\deg q - \deg p$, if we want an absolute grading, then we are required to put $2c_1(TM) = \mu_L = 0$. If not, then the grading is defined only modulo some integer dividing these two classes.

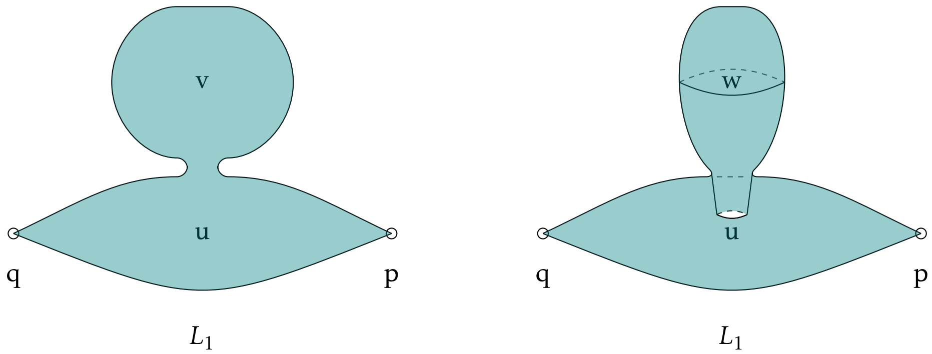


Figure 6: Obstructions to absolute grading

We notice that $2c_1(TM) = 0$ is equivalent to the bicanonical line bundle $\Lambda^n(TM)^{\otimes 2}$ being trivial. We now record the following useful fact:

Proposition 2.5 (Absolute grading): *Let $\mathcal{L} \to M$ be the fiber bundle whose fiber $\mathcal{L}_p$ consists of the linear Lagrangians in T_pM. Then TFAE:*

- *There is a global $\mu \in H^1(\mathcal{L};\mathbb{Z})$ which restricts to the universal Maslov class on each fiber $\mathcal{L}_p \simeq \mathcal{L}(n)$*
- *There is a cover $\tilde{\mathcal{L}} \to \mathcal{L}$ which restricts to the universal cover on each fibre*
- *The bicanonical bundle $\Lambda^n(TM)^{\otimes 2}$ is trivial, i.e. $2c_1(TM) = 0$.*

Even though we defined the Maslov class as a homomorphism from $\pi_2(M,L)$ to the integers, there is another related way of thinking about it. When the bicanonical bundle $\Lambda^n(TM)^{\otimes 2}$ is trivial, it admits a global section of unit length θ. This defines a phase function $\det^2_\theta : \mathcal{L} \to S^1$ and the composition

$$L \xrightarrow{s_L} \mathcal{L} \xrightarrow{\det^2_\theta} S^1$$

on fundamental groups gives us a class denoted in the same way $\mu_L = s_L^*\mu \in H^1(L;\mathbb{Z})$. Here, $s_L(p) = T_pL$ is the Lagrangian inside T_pM. This class serves as an obstruction to a grading on the Lagrangian submanifold L and when both $2c_1(TM) = 0$ and $\mu_L = 0$, there is a lift of the phase function and there is a grading, also defined using Maslov indices, such that $\mathrm{ind}([u]) =$

deg(q) − deg(p). For a detailed treatment of the grading, refer to [20].

Why does the differential square to zero? We now come to the last problem of showing that the object defined is a chain complex. The idea will be the same as in Morse theory, hence we will need a way to compactify the moduli spaces of J-holomorphic curves. However, again one needs to be careful when doing this, as the boundary strata are quite different than the ones in Morse theory. This comes down to the fact that the energy (i.e. symplectic area) of u can concentrate in three different spots, giving rise to three types of nodal degeneration. The first and most preferred case is when the energy concentrates around $s = \pm\infty$, giving rise to a phenomenon known as strip breaking. This behaves in exact analogy with how gradient flow lines break in Morse theory. The other two - disc and sphere bubbling - occur when energy concentrates either on the boundary i.e. on L_0 or L_1, or on the interior. These can be discarded once we put extra conditions, e.g. in the setting of exact Lagrangian submanifolds (which are the main subject of this essay), or something such as $\omega \cdot \pi_2(M, L_i) = 0$ for $i = 0, 1$. Notice that we need finite energy to even talk about energy concentrating in one place or another.

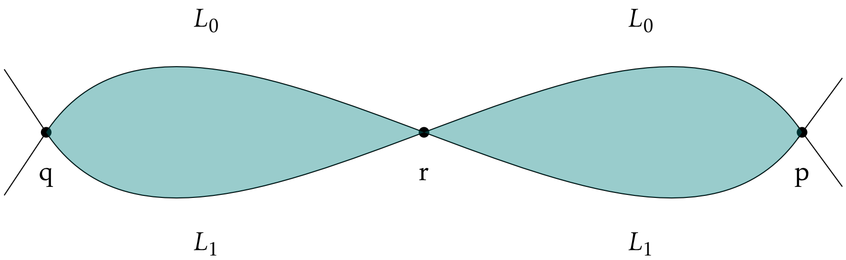


Figure 7: A broken strip

Finally, with all of this in mind, let us give the definition:

Definition 2.6 (Floer complex):

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda \cdot p \textit{ with differential}$$

$$dp = \sum_{\text{ind}[u]=1} \#\mathcal{M}(p, q; [u]) T^{\omega \cdot [u]} q$$

Here Λ is the Novikov field and we count the sum with orientation signs.

Remark: In the situation we will be dealing with, i.e. T^*S^n, which is an exact symplectic manifold, one can discard the Novikov coefficients and work straightforwardly with any coefficients.

Remark: We assumed L_0 and L_1 intersected transversely - if they do not, this can be fixed by perturbing one by a Hamiltonian isotopy, which however also perturbs the Cauchy-Riemann equation by a Hamiltonian vector field.

We run through the argument that $d^2 = 0$, which follows more or less for the same reason as in Morse homology. Given an index 2 strip u between p and q, the compactification of $\mathcal{M}(p,q;[u])$ is a compact 1-manifold, whose boundary consists of broken strips, provided no bubbling occurs (by Gromov compactness). Note that when strip breaking occurs, then we have $[u] = [u'] + [u'']$ in $\pi_2(M, L_0 \cup L_1)$, and the index is additive, so both u' and u'' have index 1. But these broken strips is precisely what d^2 is counting, and it is also the oriented boundary of a compact 1-manifold, which is 0!

Hence, we can define **Floer cohomology** $HF^\bullet(L_0, L_1)$ to be the cohomology of the cochain complex CF. A priori, this depends on the choice of J and in the context of non-transverse intersection, on the Hamiltonian perturbation H. However, the spaces of such objects are contractible and there are continuation maps between the complexes w.r.t. (H,J) and (H',J') that produce chain homotopies and hence induce an isomorphism on cohomology, and so the cohomology is independent of any choices.

2.2.4 Properties of Floer cohomology

Recall by Weinstein's theorem 1.6 that every Lagrangian has a small neighbourhood such that it is symplectomorphic to a neighbourhood of the zero section. If we consider the self cohomology $HF^\bullet(L,L)$, we need to perturb L by a Hamiltonian isotopy, but it can be made small enough so that $\psi(L)$ is within a tubular neighbourhood of L. But, for Lagrangian submanifolds, the normal bundle is isomorphic to the cotangent bundle, so we can idenfity L with the zero section and $\psi(L)$ with the graph of a closed 1-form inside T^*L. Hence, in order to compute $HF^\bullet(L,L)$ one can work entirely within the symplectic manifold T^*L. We can then cite the following result due to Floer, comparing Floer and Morse cohomology, a summary of which can be found in [4], Proposition 1.13:

Proposition 2.7 (Floer vs Morse cohomology): *Let L be embedded as the zero section inside T^*L and L' the graph of ϵdf, where f is a Morse function on L, intersecting transversely at the critical points of f. We have that for a sutiable J*

$$CF(L,L') \simeq C_{Morse}(L)$$

inducing an isomorphism on cohomology.

Combining this with the observations above, we have the following corollary:

Corollary 2.8 (Floer self-cohomology): *For an arbitrary compact Lagrangian $L \subset M$ such that $\omega \cdot \pi_2(M, L) = 0$ we have that*

$$HF^\bullet(L, L) \simeq H^\bullet(L; \Lambda)$$

This has been further generalized by Pozniak [18], Corollary 3.4.13:

Proposition 2.9 (Pozniak): *Given two Lagrangians L_0, L_1 intersecting cleanly, i.e. $N = L_0 \cap L_1$ has $TN = TL_0|_N \cap TL_1|_N$, if the triple (M, L_0, L_1) is monotone (as in definition 2.3) and $\Sigma_0, \Sigma_1 \geq 3$, $\dim N + 1 < N_\mu$, then*

$$HF_\bullet(L, L'; \mathbb{Z}/2) \simeq H_\bullet(N; \mathbb{Z}/2)$$

Remark (Pozniak-Seidel spectral sequence): In fact, Seidel [19], Section 2, shows that when $N = \coprod C_p$ is a disjoint union of its connected components, then there is a spectral sequence converging to the Floer homology $HF_\bullet(L, L')$ whose first page is

$$E^1_{pq} = \begin{cases} H_{p+q-i'(C_p)}(C_p); 1 \leq p \leq r \\ 0 \end{cases}$$

If the intersection is connected, as will be the case that we will use this result for, the spectral sequence collapses at E_2.

2.2.5 Higher operations and the Fukaya category

In the current situation, we have described a level 1 operation, i.e. an operation with one input and one output, given by the differential which satisfies $d^2 = 0$. However, there are also higher operations on the chains which give a wealth of geometric information, which we will describe now.

Firstly, there is a product, which counts pseudoholomorphic triangles with three marked points defined by:

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$$
$$p \cdot q = \sum_{ind([u])=0} \#\mathcal{M}(p, q, r; [u]) T^{\omega([u])} r$$

The moduli space $\mathcal{M}(p, q, r; [u])$ is counting J-holomorphic maps $v : (D, \partial D) \to (M, L_0 \cup L_1 \cup L_2)$ in the homotopy class of u with three marked points which are sent to p, q, r respectively, see 9. We also mod out by the automorphisms of D, i.e. the Mobius transformations.

By counting index 1 triangles and assuming no bubbling occurs, one can show, using the boundary of the moduli space of an index 1 strip, that this satisfies the Leibniz rule, i.e. the fact that μ^2 is a chain map with respect to the twisted differential on the tensor product complex:

$$d(p \cdot q) = \pm dp \cdot q \pm p \cdot dq \iff \mu^1 \circ \mu^2 = \mu^2 \circ (\mu^1 \otimes \mathbf{1} \pm \mathbf{1} \otimes \mu^1)$$

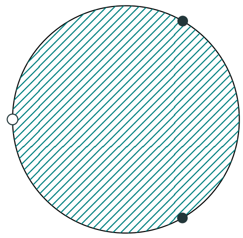


Figure 8: A disk with three marked points: 2 inputs and one output

The three terms come from three different possibilities of strip breaking occuring at the three marked points. In other words, we have a 1-dimenisonal moduli space $\mathcal{M}(p,q,r;[u])$ when the index of u is 1, whose boundary gives a nullcobordism between the three terms.

The point is that when no bubbling occurs, the J-holomorphic maps faithfully reflect the operadic structure of the domain spaces, which are the moduli spaces of disks with $k+1$ points, modulo $Aut(D)$, denoted K_k. These are called *Stasheff associahedra* and can also be thought of as planar trees. Since the Mobius transformations can fix three points, the other $k-2$ points can move freely and hence K_k is a contractible $k-2$ dimensional manifold. These come with natural compactifications, where the moving points approach the fixed points and collide with them, which we visualize by adding a zoomed in version of the disk at the collision point. Let's illustrate one of these spaces, the space K_3, which compactifies to $\overline{K_3} \simeq [0,1]$.

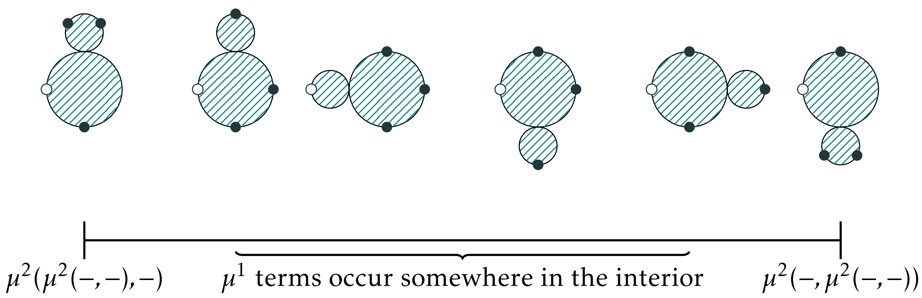


Figure 9: The compactified associahedron $\overline{K_3}$

Let's say we have fixed $-1, i, -i$ using Mobius transformations and 1 can move around. On the boundary, we get the disks where the point 1 has collided with $\pm i$. Given an index 0 strip u, the moduli space $\mathcal{M}(p,q,r,s;[u])$ has dimension $3-2+0=1$. When we project this nullcobordism $\mathcal{M}(p,q,r,s;[u])$ to the domain associahedron K_3, we get 3 separate nullcobordisms, and the boundary terms represent the terms which have two μ^2's, whereas the interior contains all the cases where μ^1 appears. In other words, the boundary represents the degenerations happening in the domain space, which is the associahedron, and the interior contains the cases where we have degenerations in the target space (e.g. strip breaking).

All in all, we get a formula which tells us that μ^2 is associative up to a homotopy given by μ^3:

$$\mu^2(\mu^2(p,q),r) - \mu^2(p,\mu^2(q,r)) = \pm\mu^3(\mu^1(p),q,r) \pm \mu^3(p,\mu^1(q),r) \pm \mu^3(p,q,\mu^1(r)) \pm \mu^1(\mu^3(p,q,r))$$

A similar argument shows that this generalizes: we have multiplication maps obeying the relations

$$\mu^k : CF(L_{k-1}, L_k) \otimes ... \otimes CF(L_0, L_1) \to CF(L_0, L_k)[2-k]$$
$$\mu^k(p_k, .., p_1) = \sum_{q \in L_0 \cap L_k, \mathrm{ind}([u]) = 2-k} \#\mathcal{M}(p_1, ..., p_k, q; [u]) T^{\omega \cdot [u]} q$$

The index is set to $2-k$ as the dimension of the moduli space turns out to be $k-2+\mathrm{ind}([u])$. By analysing the boundary of the compactification of moduli spaces of curves, one can show that this satisfies the A_∞ relations

$$\sum_{l=1}^{k} \sum_{j=0}^{k-l} (-1)^{j+\deg(p_1)+...+\deg(p_j)} \mu^{k+1-l}(p_k, ..., p_{j+l+1}, \mu^l(p_{j+l}, ..., p_{j+1}), p_j, ..., p_1) = 0$$

or more succinctly with $r = k - l - j$

$$\sum_{k=r+l+j, l>0} \pm \mu^{r+j+1}(\mathbf{1}^{\otimes r} \otimes \mu^l \otimes \mathbf{1}^{\otimes j}) = 0$$

In particular, the product descends to a well-defined, associative operation on cohomology. All of this information can be packaged into the so called **Fukaya category**:

Definition 2.10 (Fukaya category): *For (M, ω) a symplectic manifold with some extra properties, one can consider the A_∞-category* Fuk(M) *such that:*

- Ob Fuk(M) *consist of compact, closed, oriented, spin (i.e. $w_1(L) = w_2(L) = 0$) Lagrangians which don't admit bubbling (e.g. $\omega \cdot \pi_2(M, L) = 0$) and with vanishing Maslov class.*
- Mor$(L_0, L_1) = CF(L_0, L_1)$ *with appropriately chosen perturbation data (see [23]), so that they can be moved to intersect transversely.*

There is a useful refinement of the above definition in which we equip each Lagrangian L with a flat complex vector bundle (flatness is required so that parallel transport is independent of homotopy classes). More precisely, we put

$$CF((L, \mathcal{E}), (L', \mathcal{E}')) = \bigoplus_{p \in L_0 \cap L_1} \hom(\mathcal{E}|_p, \mathcal{E}'|_p)$$

Given two intersections q, p, with p an input and q an output, we need to define the coefficients in $\mu^1(x)$ for $x \in \hom(\mathcal{E}|_p, \mathcal{E}'|_p)$ which requires to find an element in $\hom(\mathcal{E}|_q, \mathcal{E}'|_q)$. We do this as follows: first, parallel transport $\mathcal{E}$ along the boundary of u from q to p, then apply x and finally parallel transport again:

$$\eta_{[u],p} : \mathcal{E}|_q \xrightarrow{p.t.} \mathcal{E}|_p \xrightarrow{x} \mathcal{E}'|_p \xrightarrow{p.t.} \mathcal{E}'|_q$$

Then we can define

$$\mu^1(x) = \sum_{q \in L \cap L', \mathrm{ind}\,[u] = 1} \#\mathcal{M}(p, q; [u]) T^{\omega \cdot [u]} \eta_{[u],p}$$

In a similar vein one defines the higher operations.

Remark: An A_∞ category can be seen as a higher homotopical generalization of a dg category, i.e. a category with a differential and a product satisfying the Leibniz rule. In other words, a dg category is an A_∞ category with $0 = \mu^3 = \mu^4 = ...$ A canonical example is Ch, the category of chain complexes over a field $\mathbb{K}$, which has Hom sets the chain complex whose n-th piece is

$$\mathrm{Hom}^n(A_\bullet, B_\bullet) := \prod_i \mathrm{Hom}(A_i, B_{i+n})$$

$$(df)_i = d^B f_i + (-1)^{n+1} f_{i-1} d^A \in \mathrm{Hom}^{n-1}(A_\bullet, B_\bullet)$$

It is easy to see tha $d^2 = 0$ and that the closed morphisms are precisely the chain maps.

2.2.6 The infinitesimal Fukaya category of a cotangent bundle

Up until now, our definition of the Fukaya category consisted of compact Lagrangians. However, in the case of cotangent bundles T^*M, we would like to consider non-compact exact Lagrangians with nice behaviour at infinity, as for example a cotangent fibre. There are different approaches to doing this - one is the infinitesimal Fukaya category of Nadler-Zaslow [17], and the other is the wrapped Fukaya category of Abouzaid and Seidel. In chapter 4, we will be using the former, and now give a very brief outline of the objects and morphism groups, following [17] and [9].

Let $\omega = d\lambda$ be the canonical exact symplectic form on T^*M. This carries a Liouville vector field Y, the radial rescaling vector field, dual to λ, i.e. $\omega(Y, -) = \lambda$. The magic formula then tells us that

$$\mathcal{L}_Y \omega = d\iota_Y \omega + \iota_Y d\omega = d\lambda = \omega$$

We consider exact Lagrangians, i.e. $L \subset T^*M$ such that $\lambda|_L$ is exact. Moreover, we require that $\overline{L}$ is a subanalytic subset of the compactification $\overline{T}^*M$. This means that the Lagrangian is "Legendrian" at infinity. To define the morphism groups $CF(L_0, L_1)$, we need to modify the Lagrangians so that L_0 and L_1 intersect transversely in T^*M and do not intersect at all at infinity. To do this, we need to introduce a very small perturbation of L_0 in a specified direction. This is done by choosing a function $H : T^*M \to \mathbb{R}$, for example a function that outside of a compact subset is $H(x, \xi) = |\xi|$, which generates a Hamiltonian isotopy ϕ_H that is the normalized geodesic flow outside a compact subset. Hence, we put

$$CF(L_0, L_1) := CF(\phi_H(L_0), L_1)$$

A technical issue arises when we consider composition maps: what we actually get is

$$CF(L_1, L_2) \otimes CF(L_0, L_1) := CF(\phi_{H_{12}} L_1, L_2) \otimes CF(\phi_{H_{12}} \phi_{H_{01}} L_1, \phi_{H_{12}} L_2) \to CF(\phi_{H_{12}} \phi_{H_{01}} L_0, L_2)$$

However, we would like to land in $\phi_{H_{02}} L_0$ instead - to fix this, one has to make the functions H sufficiently small, so that the result is an isomorphism on the cochain groups.

Note also that the bicanonical bundle of T^*M is trivial, which is needed for the purpose gradings.

3 The Dehn twist, algebraically and geometrically

In this section, we will define the Dehn twist by recourse to Picard-Lefschetz theory, a holomorphic analogue of Morse theory. Moreover, we will define and state various properties of Lefschetz fibrations and utilize Seidel's topological quantum field theory to show that the effect of the Dehn twist on the Fukaya category fits into an exact triangle. This will result in Seidel's long exact sequence in Floer cohomology.

3.1 Picard-Lefschetz theory, Lefschetz fibrations and the model Dehn twist

We now consider holomorphic maps from a complex manifold to $\mathbb{C}$ and the critical points of such maps will give us important geometric information, but in a slightly different way than in Morse theory. Recall that the homotopy type of the manifold M changes precisely when one hits a critical point, and the critical values separate $\mathbb{R}$ into disconnected components (intervals). In the complex case, removing the critical values won't disconnect it, but the analogue of hitting a critical point becomes traversing a loop around a critical value. The change in the topology of the fibers will be reflected by the monodromy, and the main point is that it is given by Dehn twisting by a vanishing cycle. We begin with the local theory, where we have an analogue of the smooth Morse lemma:

Lemma 3.1 (Complex Morse lemma): *If X is a complex variety and p is a non-degenerate critical point (w.r.t. the complex Hessian), then there are holomorphic coordinates in a neighbourhood of p such that*

$$f = f(p) + \sum z_i^2$$

For a proof, see [27]. Note that in this situation, there is no invariant like the index. This lemma shows that there is a standard example, to which most computations eventually reduce to.

Example (Canonical example): Consider the map $\pi : \mathbb{C}^n \to \mathbb{C}, (z_1, \ldots, z_n) \mapsto \sum z_i^2$. This has a single critical point 0. To be able to compute the monodromy, we need some connection on $\mathbb{C}^n$. To do this, we equip $\mathbb{C}^n$ with its standard symplectic form

$$\omega = \sum dx^i \wedge dy^i = \frac{i}{2} \sum dz^i \wedge d\bar{z}^i$$

Then, the tangent spaces away from the critical points split as follows:

$$T_z\mathbb{C}^n = \mathrm{Vert}_z \oplus \mathrm{Hor}_z$$

$$\mathrm{Vert}_z = \ker d\pi_z, \ \mathrm{Hor}_z = \mathrm{Vert}_z^{\perp}$$

The complement is given with respect to ω:

$$\mathrm{Hor}_z = \{v \in T_z\mathbb{C}^n | \omega(v, w) = 0 \, \forall w \in \mathrm{Vert}_z\}$$

Example (Continued): In fact, this procedure can be done for any symplectic fibration and produce an Ehresmann connection. In our example, the horizontal subspace will be given by $\mathrm{Hor}_z = \mathbb{C}\overline{z}$.To see this, note that $w \in \ker d\pi_z \iff \sum w_i z_i = 0$. The set of these form an $n-1$ dimensional subspace of $T_z\mathbb{C}^n$ and its complement consists of v such that $\omega(v,w) = 0$. But this is equivalent to

$$\sum v_i \overline{w}_i - w_i \overline{v}_i = 0 \text{ for any } w \text{ s.t.} \sum w_i z_i = 0$$

We clearly see that $v = \overline{z}$ is a solution and must in fact be the "only" one, since the complement is 1-dimensional.

Now, consider the loop parametrizing the unit circle $\gamma(t) = e^{2\pi i t}$. The monodromy is, by definition, the time 1 value of a horizontal lift of γ :

$$\begin{array}{ccc} & & \mathbb{C}^n \\ & \overset{\gamma^\#}{\nearrow} & \downarrow \pi \\ [0,1] & \xrightarrow[\gamma]{} & \mathbb{C} \end{array}$$

It must satisfy $\dot{z} = \dot{\gamma}^\#(t) = \lambda_t \overline{z}$ for some $\lambda_t \in \mathbb{C}$ since it is horizontal and also $\pi(\gamma^\#(t)) = \sum \gamma_i^\#(t)^2 = e^{2i\pi t}$ since it lifts γ. Differentiating both sides, we get

$$\frac{d}{dt}[\sum \gamma_i^\#(t)^2] = 2\sum \dot{\gamma_i}^\#(t)\gamma_i^\#(t) = 2\pi i e^{2\pi i t}$$

$$2\lambda_t \sum \overline{z}_i z_i = 2\pi i e^{2\pi i t}$$

$$\lambda_t = \frac{i\pi e^{2\pi i t}}{|z|^2}$$

In other words, the lift satisfies the differential equation

$$\dot{z} = \frac{\pi i e^{2\pi i t}}{|z|^2}\overline{z}$$

We can see that $|z|^2$ is constant for such solutions, as its derivative has

$$\frac{d}{dt}\langle z,z\rangle = \langle \dot{z},z\rangle + \langle z,\dot{z}\rangle = \frac{i\pi}{|z|^2}\Big[e^{2\pi i t}\langle \overline{z},z\rangle - \overline{e^{2\pi i t}\langle \overline{z},z\rangle}\Big]$$

But

$$e^{2\pi i t}\langle \overline{z},z\rangle = e^{2\pi i t}\sum \overline{z}_i \overline{z}_i = 1$$

since $\pi(z) = e^{2\pi i t}$, hence $\frac{d}{dt}|z|^2 = 0$ and it must be constant. This allows us to reduce the question to a linear ODE by putting $f = e^{-2i\pi t}z$. After solving this, we get that the monodromy is determined by the equality

$$|y(1)|x(1) + i|x(1)|y(1) = -\exp(-\frac{2\pi i|x(0)||y(0)|}{|z(0)|^2})(|y(0)|x(0) + i|x(0)|y(0))$$

where $z(t) = x(t) + iy(t)$

Now we introduce the next concept - a Lefschetz fibration is informally a family of hypersurfaces with only a finite number of critical points, which are modeled on the above situation, the simplest type of singularity in algebraic geometry (the ordinary double point).

Definition 3.2 (Symplectic Lefschetz fibrations): *An exact symplectic fibration with singularities consists of the following data:*

- *A map $\pi : E \to B$ between exact symplectic manifolds which is pseudoholomorphic with respect to the almost complex structures.*
- *π is transverse to the boundary of B i.e. for all $x \in \pi^{-1} \in C \subset \partial B$ we have $d\pi(T_x E) + T_{\pi(x)}(C) = T_{\pi(x)} B$. Thus $\pi^{-1}\partial B := \partial^v E$ is a boundary stratum, the vertical boundary, which has complement $\partial^h := \partial E - \partial^v E$.*
- *$\pi_{\partial^h E}$ is a smooth fibration. Moreover, if $x \in \partial^h E$ then $(T_x E)^h \subset T_x(\partial^h E)$, where the left side is the horizontal subspace orthogonal to $\ker d\pi$ defined via the connection given by the global form ω on E.*

An exact ***Lefschetz fibration*** *over a compact Riemann surface $\pi : E \to \Sigma$ consists of a map π such that the critical points of π are nondegenerate and locally integrable, i.e. π has the local form as described in the canonical local model: $\pi = \pi(p) + \sum z_i^2$.*

The importance of Lefschetz fibrations to symplectic geometry was first noticed by Arnol'd in his paper [1]. The point is that the nonsingular fibers of the local model can naturally be identified with symplectic manifolds, the cotangent bundles T^*S^{n-1}, and the monodromy is a symplectomorphism, which is called a **Dehn twist**. Here's a picture which illustrates the case $n = 2$:

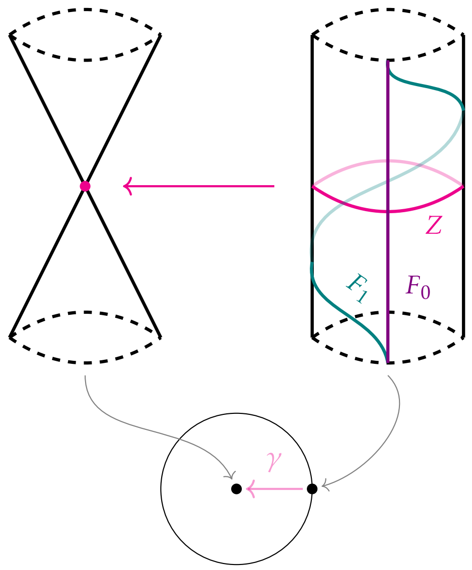


Figure 10: The singular fibre (left) and a generic fibre (right)

The cylinder on the right is $\pi^{-1}(1)$ which consists of the points

$$\pi^{-1}(1) = \{x + iy | |x|^2 - |y|^2 = 1, \langle x, y \rangle = 0\}$$

On the other hand, the tangent, and hence cotangent bundle, can be calculated by differentiating the defining equality $\langle u, u \rangle = 1$ of the sphere to get

$$T^*S^{n-1} = \{u + iv | \, |u|^2 = 1, \langle u, v \rangle = 0\}$$

These two manifolds are symplectomorphic via $(x,y) \mapsto (\frac{x}{|x|}, |x|y)$. The Dehn twist, following the computation 3.1, takes the shape of the symplectomorphism which obeys the following equation, where $|v| = |v'|$[3]:

$$\psi(u+iv) = u' + iv', u' + i\frac{v'}{|v'|} = -\exp(-\frac{2\pi i|v|}{\sqrt{1+4|v|^2}})(u + i\frac{v}{|v|})$$

One can see that when the cotangent coordinate v is very big, the Dehn twist is close to the identity, whereas when v is close to 0, the effect is the antipodal map. In the picture 10 we have that $\psi(F_0) = F_1$.

Another equivalent way to view the Dehn twist[4] is to consider the Hamiltonian function $H : T^*S^{n-1} \to \mathbb{R}, H(p,q) = |p|$, where p is the fiber coordinate. This generates the normalized geodesic flow σ away from the zero section, which flows a vector along the geodesic emanating from it at unit speed. We can rescale H by a suitable function $r : \mathbb{R} \to \mathbb{R}$ to make the speed depend on the size of the vector. We require that r is 0 for large enough t, and moreover $r(-t) = r(t) - t$, and then $H(r(|p|))$ will generate an isotopy $\phi_t(p,q) = \sigma_{tr'(|p|)}(p,q)$. Since $r'(0) = 1/2$, the time 2π map can be extended via the antipodal map on the zero section and defines a generalized Dehn twist:

$$\tau(p,q) = \begin{cases} \sigma_{2\pi r'(|p|)}(p,q), \; p \neq 0 \\ (0,-q), p = 0 \end{cases}$$

Different choices of r will produce different formulas, but will result in Hamiltonian isotopic Dehn twists.

Now we come back to the canonical fibration $\pi : \mathbb{C}^n \to \mathbb{C}$ and relate the Dehn twist with monodromy. Notice that over the singular point 0 (see 10) the fibers degenerate to a nodal singularity, and in the picture we can see that the zero-section $S^{n-1} \subset T^*S^{n-1}$, which corresponds to the real slice in $\pi^{-1}(1)$, become smaller and smaller until they shrink to a point. This forms a Lagrangian submanifold of each generic fibre and bears the name of a **vanishing cycle**. Here is an important theorem concerning vanishing cycles:

Theorem 3.3 (Vanishing cycles and Dehn twists): *Let $E \to B$ be a symplectic Lefschetz fibration. Given a path γ with $\gamma(1) = c$, but not hitting any other critical point, then the vanishing cycle V_γ is defined as the set of points in the fibers which parallel transport to the singularity of $\pi^{-1}(c)$. This is a Lagrangian sphere and the monodromy of a loop encircling c is given, up to isotopy, by the Dehn twist around V_γ. Different loops may produce different symplectomorphism, but they are all Hamiltonian isotopic.*

A proof of this can be found in Proposition 1.15, [21]. Using the Weinstein neighbourhood theorem, every time we have a Lagrangian sphere we can create generalized Dehn twists, by generalizing the local model to any exact symplectic manifold ([21], Proposition 1.11).

[3]This is all taken from [15], page 269

[4]This definition is taken from Seidel's lectures on 4-dimensional Dehn twists [24]

Proposition 3.4 (Standard fibration): *Whenever $L \subset M$ is an exact Lagrangian sphere inside of an exact symplectic manifold, then there is a standard fibration $E^L \to D$ over the disk whose fibres are symplectomorphic to M and such that the monodromy is given by a Dehn twist along L.*

All in all, we have seen that Dehn twists arise from monodromies around critical points, and that the converse is also true - the monodromy around a critical point is given by a Dehn twist along a vanishing cycle.

3.2 Pseudoholomorphic sections, relative Gromov-Witten maps and Seidel's TQFT

Up until now, we have been counting pseudoholomorphic maps from $I \times \mathbb{R}$ to a target symplectic manifold, obeying Lagrangian boundary conditions. This allowed us to define Lagrangian Floer cohomology.

However, this can be seen as a particular example of counting pseudoholomorphic sections: namely, if we consider the trivial fibration $M \times T \to T$ where $T = I \times \mathbb{R}$, then a certain set of sections with Lagrangian boundary counditions exactly recover the same moduli space.

More generally, instead of thinking about pseudoholomorphic maps from the strip $I \times \mathbb{R}$, we can consider fibrations $E \to S$ over a surface with "strip-like ends" and count pseudoholomorphic sections. This will result in relative invariants, i.e. maps between the Floer chain groups, just like the product μ^2.

We now present the definitions of these objects and lay out some of their properties. There is a lot of underlying technical machinery that needs to be set up for all of these things to work, but we refer to [21] for the details.

3.2.1 Definition of the relative invariants

Definition 3.5 (Moduli of sections): *Given an exact symplectic Lefschetz fibration $E \to \Sigma$ with fibers symplectomorphic to M, we define a Lagrangian boundary condition to be a half-dimensional submanifold $F \subset E$ contained in $\pi^{-1}\partial\Sigma$ such that each F_z is an exact Lagrangian submanifold of $E_z \simeq M$ which parallel transport into each other. A pseudoholomorphic section $u : \Sigma \to E$ subject to the boundary condition is a J-holomorphic section such that $u(\partial\Sigma) \subset F$. We define $\mathcal{M}_{E/\Sigma}$ to be the moduli space of such sections, which can be thought of as the zero set of a section $\overline{\partial}$ of an infinite Banach bundle, just as in the usual case. The zero stratum $\Phi_{E/\Sigma}$ of isolated points of this moduli space consists of a finite set of points.*

Example (Local model): Let's consider the restriction E of the local model $\mathbb{C}^{n+1} \to \mathbb{C}$ to the unit disk D, with Lagrangian boundary given by the real slices, corresponding to the zero sections under the symplectomorphism to T^*S^n. The pseudoholomorphic sections then amount to holomorphic sections $u : D \to E$ with the boundary condition $\mathrm{im} u_{\partial D} \subset \sqrt{z} S^n$, in other words by projecting we get holomorphic maps

$$u_i : D \to E \xrightarrow{p_i} \mathbb{C}$$

which obey $u_i(z) \in \sqrt{z}\mathbb{R}$ for $z \in S^1$. By looking at imaginary parts, it is easy to see that this condition is equivalent to $u_i = \overline{u_i} z$ on S^1 and hence by looking at the expansion $u_i(z) = \sum a_i z^i, \overline{u_i(z)} = \sum \overline{a_i z}^i = \sum \overline{a_i} z^{-i}$ on S^1 and comparing coefficients, we get that $a_k = 0, k > 1$ and $a_0 = \overline{a_1}$. Hence, the function $\tilde{u_i} = u_i - (a_0 + \overline{a_0} z)$ is holomorphic and equal to 0 on the boundary ∂D, where it achieves its maximum, hence must be identically zero. All in all, we must have that $u(z) = az + \overline{a}$. The fact that u is a section, along with the boundary conditions, then imply that a must satisfy $\|a\|^2 = 1/2, \sum a_i^2 = 0$. This amount to three equations on the variables $x_1, y_1, ..., x_{n+1}, y_{n+1}$ which give the real and imaginary parts of a and hence the dimension is $2n + 2 - 3 = 2n - 1$. This space has no isolated points, so in this case $\Phi = 0$.

This local computation also generalizes to show that the standard fibration 3.4 also has trivial Φ.

We now define the relative Gromov-Witten invariants, which arise as counts of pseudoholomorphic sections which have fixed asymptotic behaviour at the strip-like ends and have Lagrangian boundary conditions.

Definition 3.6 (Relative invariants): *Consider a symplectic Lefschetz fibration $E \to \Sigma$ over a compact Riemann surface with a finite set of marked points on the base e_i^+, e_i^- around which the fibration is trivial. Let $S = \Sigma \setminus \{e_i^\pm\}$ be the surface with strip-like ends where each e gets replaced with a little strip $\epsilon_e \simeq T \simeq I \times \mathbb{R}$ and which comes with trivializations $T^\pm \times M \to E$ over each such end. Given a Lagrangian boundary condition F and using this local triviality near the strips, we get pairs of Lagrangians $F_{\epsilon_{s,k}} = L_{e,k}, k = 0, 1$ for each marked point e i.e. strip-like end ϵ_e. This allows us to define maps on the chain groups*

$$C\Phi_{E/S} : \bigotimes CF(L_{e^+,0}, L_{e^+,1}) \to \bigotimes CF(L_{e^-,0}, L_{e^-,1})$$
$$\otimes y_{e^+} \mapsto \sum \Phi_{E/S}(y_{e^-}, y_{e^+}) \otimes y_{e^+}$$

Amazingly, doing this fits into the framework of a topological quantum field theory: the idea is that whenever we have fibrations over two such surfaces with strip-like ends, we can glue the bases and bundles together so that gluing on the topological level corresponds to composition of relative maps on the algebraic level.

For example, recall that the original definition of the product μ^2 used counts of pseudoholomorphic triangles with three marked points, which corresponds to a pair of pants surface, as in 11. (the inputs are on the right as we want to work with cohomology).

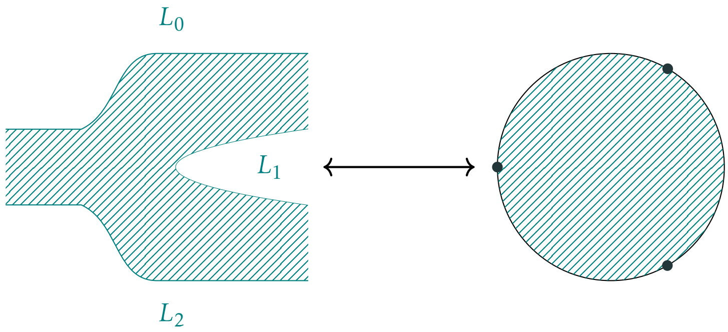


Figure 11: Pairs of pants correspond to counting pseudoholomorphic triangles with three marked points

The relative invariant (for the trivial fibration) then is precisely

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \xrightarrow{\mu^2} CF(L_0, L_2)$$

Similarly, the unit in $HF^\bullet(L, L)$ occurs as the relative invariant associated to the unit half-disk, i.e. the map with a trivial input and one output and a single Lagrangian boundary condition L.

Let's see what happens when we glue two surfaces. The TQFT axioms state that we should get an invariant associated to the bundle $(E, S = S_1 \# S_2)$ which is glued from the two bundles $(E_1, S_1), (E_2, S_2)$ and moreover that the two are chain homotopic, hence induce the same maps on Floer cohomology:

$$C\Phi_{E/S} \sim C\Phi_{E_2/S_2} \circ C\Phi_{E_1/S_1}$$

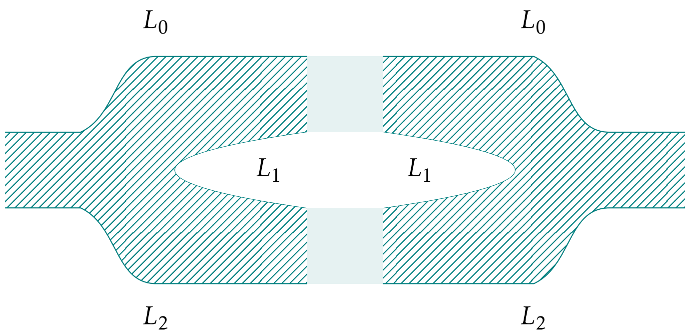


Figure 12: Gluing two surfaces with strip-like ends

In the situation from the picture above, this means that

$$C\Phi_{E/S} : CF(L_0, L_2) \to CF(L_0, L_2)$$
$$CF(L_0, L_2) \xrightarrow{C\Phi_{E_2/S_2}} CF(L_1, L_2) \otimes CF(L_0, L_1) \xrightarrow{C\Phi_{E_1/S_1}} CF(L_0, L_2)$$

Given $p \in L_0 \cap L_2$, the coefficient of s is counting ways to get from p to s via two pseudoholomorphic triangles (p, q, r) and (q, r, s) interrupted by the Lagrangian L_1.

Remark (Families of fibrations give chain homotopies): Strictly speaking, the definition of the relative invariants depends on the choice of almost complex structure. However, [21], Lemma 2.30 shows that the resulting invariants are chain homotopic, so agree on cohomology. Another important feature of this TQFT is that whenever we have a smooth family of fibrations with strip-like ends $E^r \to S^r, r \in R$, we get relations of the sort $\partial \circ (map) + (map) \circ \partial + (boundary\, terms)$ and in particular when we have a 1-parameter family, we get a chain homotopy h between the relative invariants on both ends:

$$\partial h + h\partial = C\Phi_1 - C\Phi_2$$

This fact will be extremely useful in the next section.

3.3 Seidel's long exact sequence in Floer cohomology

Now we give a sketch proof of the long exact sequence in Lagrangian Floer cohomology, following Seidel's original paper [21]. Let Z be an exact Lagrangian sphere, which we can Dehn twist along. Given a Lagrangian A we want to prove that there is an exact triangle:

$$\begin{array}{ccc} & A & \\ \nearrow & & \searrow \\ HF^\bullet(Z,A) \otimes Z & \longleftarrow & \tau_Z(A) \end{array}$$

What this means is that we have a long exact sequence in cohomology when we Hom into any other test Lagrangian B (which is contravariant, so we have to flip the arrows):

$$\begin{array}{ccc} & HF^\bullet(A,B) & \\ \swarrow & & \nwarrow \\ HF^\bullet(Z,A) \otimes HF^\bullet(Z,B) & \longrightarrow & HF^\bullet(\tau_Z(A),B) \end{array}$$

To do this, using the relative Gromov-Witten invariants we will define chain maps

$$CF(Z,B) \otimes CF(\tau_Z(A),Z) \to CF(\tau_Z(A),B) \to CF(A,B)$$

We will show that the effect on cohomology fits into a long exact sequence, using an algebraic lemma (Lemma 2.32 in [21]). However, note that this does not give the same long exact sequence as the desired one: what we get is

$$\begin{array}{ccc} & HF^\bullet(A,B) & \\ \swarrow & & \nwarrow \\ HF^\bullet(\tau_Z(A),Z) \otimes HF^\bullet(Z,B) & \longrightarrow & HF^\bullet(\tau_Z(A),B) \end{array}$$

To finish up, we need to use the Poincare duality isomorphism, along with a Dehn twist:

$$HF^\bullet(\tau_Z(A), Z) \simeq HF^\bullet(\tau_Z(A), \tau_Z(Z)) \simeq HF^\bullet(A, Z) \simeq HF^\bullet(Z, A)$$

Note also that the Dehn twists changes the grading by $1-n$, as is shown in [20]. Moreover, the left pointing diagonal map is a pair of pants coproduct, whereas the lower map is a product.

3.3.1 Definition of the maps and a sketch of the main argument

The idea is to use particularly nice fibrations that we can work with, and then use a vanishing result. The whole point of introducing Dehn twists as monodromies of Lefschetz fibrations will then become apparent.

The first fibration will be a trivial one, denoted $E^b \to S^b$, producing a pair of pants product b:

$$CF(Z, B) \otimes CF(\tau_Z(A), Z) \xrightarrow{b=\mu^2} CF(\tau_Z(A), B)$$

This appears as the right hand surface in figure 13.

The second fibration, which we denote $E^c \to S^c$ is a bit trickier. We first take the standard fibration over the disk with radius 1/2, fiber M and monodromy τ_Z, then slightly modify it so that we can glue it to the fibration whose base is $\mathbb{R} \times [-1, 1] \setminus D(1/2)$ and having Lagrangian boundary conditions B on the top and interpolating between $\tau_Z(A)$ to A on the bottom from the positive to the negative side. Once these are glued together, we obtain the usual strip $S^c = \mathbb{R} \times [-1, 1]$ with Lagrangian boundary conditions as in the left hand side of 13. The circle signifies the monodromy around the critical value, which gives the Dehn twist. The map we get is

$$C\Phi_{E^c/S^c} : CF(\tau_Z(A), B) \xrightarrow{c} CF(A, B)$$

When we glue the two fibrations together $(E, S) = (E^c \# E^b, S^c \# S^b)$, the TQFT axioms state that the resulting invariants should be chain homotopic:

$$CF_{E/S} \sim CF_{E^c/S^c} \circ CF_{E^b/S^b} = c \circ b$$

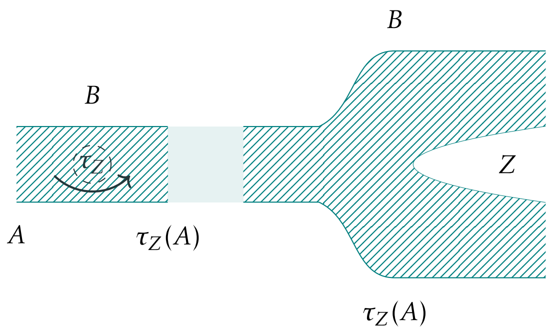


Figure 13: The two surfaces, glued together

The final step is to show that this invariant vanishes, which follows from the geometric arguments in [21]. To conclude, we invoke the following algebraic lemma, again from [21]:

Lemma 3.7 (Seidel's algebraic lemma): *Take three $\mathbb{R}$-graded vector spaces C', C, C'', each of them with a differential of order $(0;\infty)$. Suppose that we have differential maps $b : C' \to C, c : C \to C''$ and a homotopy $h : C' \to C''$ between $c \circ b$ and the zero map, such that that the following conditions are satisfied for some $\epsilon > 0$:*

- *C', C'' have gap $(0;3\epsilon)$, and C has gap $(0;2\epsilon)$.*
- *For all $r \in supp(C')$ and $s \in supp(C''), |r - s| \geq 4\epsilon$.*
- *One can write $b = \beta + (b - \beta)$ with β of order $[0;\epsilon)$ and $(b - \beta)$ of order $[2\epsilon;\infty)$; and $c = \gamma + (c - \gamma)$ with the same properties. The low order parts (which do not need be differential maps) fit into a short exact sequence of vector spaces*

$$0 \to C' \xrightarrow{\beta} C \xrightarrow{\gamma} C'' \to 0$$

- *h is of order $[0;\infty)$.*

Then the maps on cohomology induced by b, c fit into a long exact sequence

$$H^\bullet(C') \xrightarrow{b_*} H^\bullet(C) \xrightarrow{c_*} H^\bullet(C'') \to H^\bullet(C')$$

4 Exact Lagrangians in T^*S^n

Following Seidel's paper [22], we are going to use the machinery of Fukaya categories to prove something about Lagrangians inside T^*S^n. Ideally, one would like to show that any exact Lagrangian is Hamiltonian isotopic to the zero section - this is the statement of the famous **Nearby Lagrangian Conjecture**. This can be proved in the case $n = 1$ and hence we will be concerned with $n > 1$. In its current state, the full Nearby Lagrangian Conjecture seems out of reach - we will prove the weaker statement that any such Lagrangian L becomes isomorphic to the zero section Z in a triangulated extension of the Donaldson-Fukaya category. This then has the following list of consequences:

- $[L] = \pm[Z]$
- $H^\bullet(L;\mathbb{C}) \simeq H^\bullet(S^n,\mathbb{C})$
- Any two exact Lagrangians (satisfying some extra conditions) intersect nontrivially

4.1 Preliminaries and basic results

As in any paper on Floer theory, we will begin with a list of assumptions. Firstly, let F_0 be a cotangent fiber. We will be working in the infinitesimal Fukaya category as in the end of Chapter 2, and the Lagrangians we consider will have the following properties, so that the Fukaya category is as nice as possible:

- L is either compact, or agrees with F_0 almost everywhere.
- $\omega \cdot \pi_2(M,L) = 0$ to ensure no bubbling occurs.
- L is oriented and spin, i.e. $w_1(L) = w_2(L) = 0$, to ensure the moduli spaces of J-holomorphic curves obtain an orientation.
- $\mu_L = 0$ and moreover a choice of grading on L given by a choice of lift of the phase function $L \to S^1$ (see 1.9, and the paper [20])

This allows the Floer cohomology groups to be defined and, crucially, have a grading.

We can also equip the Lagrangians with branes, i.e. flat complex vector bundles ξ as in the end of section 2, 2.10.

Remark (Cohomology with local coefficients): Recall that flat bundles $\mathcal{E}$ correspond to representations of the fundamental group, i.e. a homomorphism $\rho : \pi_1(X) \to \mathrm{End}(A)$. Hence, we can define cohomology with local coefficients by considering the chains on the universal covering space $\tilde{X}$, which has a π_1 action given by deck transformations:

$$H^\bullet(X;\mathcal{E}) := H^\bullet(\mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(C_\bullet(\tilde{X}), A)$$

We will mostly be working with the trivial flat bundle, so this tool will not be crucial.

We will show that the Lagrangians satisfying the list of conditions all essentially come from only two simple Lagrangians - the fibre F_0 and its Dehn twist along the zero section $\tau_Z F_0 := F_1$. In fancier terminology, F_0 and F_1 generate the Fukaya category.

To begin with, we need to understand our three generating Lagrangians and their Floer cohomology. This comes in the form of the following lemma:

Lemma 4.1 (Floer cohomology of the three Lagrangians):

$$HF^\bullet(F_0, F_1) \simeq H^\bullet(S^{n-1}, \mathbb{C})$$

$$HF^\bullet(F_1, F_0) = 0$$

$$HF^\bullet(Z, F_i) \simeq HF^\bullet(F_i, Z) \simeq \mathbb{C}$$

Sketch of proof: Firstly, since Z and F_i intersect transversely at a single point, the Floer chain complex will have only one entry and hence the last isomorphism follows. For the Floer cohomology groups of F_0 and F_1 we need to make them transverse, as currently they agree outside of a compact subset. This can be done by moving the first one by a normalized geodesic flow: if we do this to F_1 we end up as in the left hand side of 14, which makes the Lagrangians disjoint, hence the Floer complex is trivial, and the second line isomorphism follows. Finally, the effect of moving F_0 will result in a picture like the right hand side of 14, where they intersect cleanly along S^{n-1}. Thus, we can invoke 2.9 to conclude the last isomorphism.

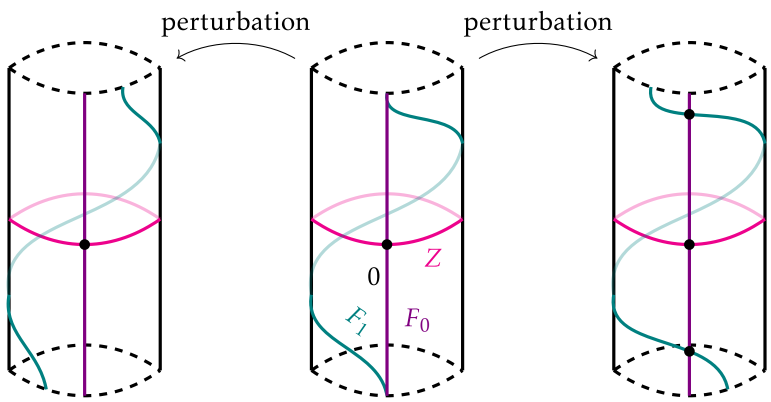


Figure 14: Perturbing the Lagrangians to intersect transversely

4.2 Black magic using triangulated categories

We would like to show that any Lagrangian L is essentially some combination (using Dehn twists, or some form of Lagrangian surgery) of the cotangent fibre F_0. Miraculously, it turns out that there is an algebraic approximation of these geometric notions - the idea of an exact triangle in a triangulated category. However, $\mathrm{Fuk}(M)$ is not triangulated, so we need to work in some form of triangulated extension. Abstractly, one can consider the Yoneda embedding, or restrict oneself to the simpler model of twisted complexes, which provides an enlargement of the Fukaya category where formal direct sums are allowed. More precisely, one can first embed $\mathrm{Fuk}(M)$ inside $\mathrm{Tw\,Fuk}(M)$, which is an A_∞ triangulated category, and then take cohomology to get an honest triangulated category which contains the Donaldson-Fukaya category: $H^0\mathrm{Fuk}(M) \subset H^0\mathrm{Tw\,Fuk}(M)$. We call this the **derived Fukaya category** and denote it by $D^b\mathrm{Fuk}(M)$.

> *Remark (Mirror symmetry):* The derived Fukaya category plays an important role in Kontsevich's homological mirror symmetry conjecture - roughly, it states that string theory has 2 models, the A-model and B-model, corresponding to studying Lagrangian intersections inside symplectic manifold versus complex subvarieties of a complex algebraic variety. Certain symplectic and complex manifolds come in mirror pairs, and the homological mirror symmetry conjecture states that the derived Fukaya category of one is equivalent to the derived category of coherent sheaves of its mirror. Moreover, under this duality, the Dehn twist, which we mention below, has an analogue on the algebraic geometry side given by spherical twist functors.

Recall that in any triangulated category, given a morphism $X \xrightarrow{a} Y$ there is an object, unique up to isomorphism (but maybe not unique isomorphism, which prevents this from being a functorial operation) denoted by $\mathrm{cone}(a)$ such that $X \to Y \to \mathrm{cone}(a)$ is distinguished. Now let us consider the following morphisms in our additive enlargement $\mathrm{Tw\,Fuk}T^*S^n$:

$$HF^\bullet(L,L') \otimes L \xrightarrow{ev} L'$$
$$L \xrightarrow{ev'} HF^\bullet(L,L')^\vee \otimes L'$$

The map is the tautological evaluation map: given $c \in HF^\bullet(L,L')$ represented by come intersection point, the map is literally c as in the usual Fukaya category. In other words, ev is a family of maps, one for each generator of the vector space $HF^\bullet(L,L')$. We will denote the cone of this map by $T_L(L')$ and similarly we define $T'_{L'}(L) = \mathrm{cone}(ev')[-1]$ (note that we have switched the L and L'). As was shown in chapter 3 following Seidel's paper [21], this turns out to recover the geometric concept of the Dehn twist!

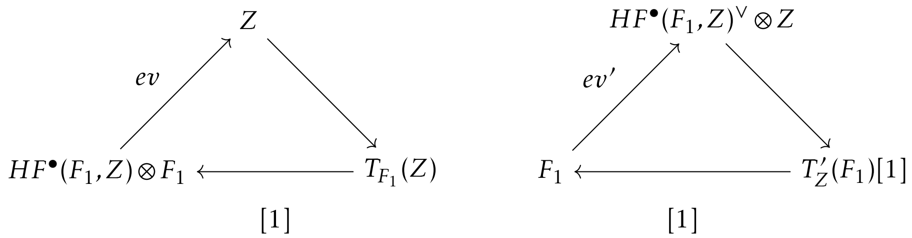
Theorem 4.2 (Algebraic twist = Dehn twist): *For any L in $D^{\flat}\mathrm{Fuk}(M)$ we have that*

$$T_Z(L) \simeq \tau_Z(L)$$
$$T'_Z(L) \simeq \tau_Z^{-1}(L)$$

This allows us to prove some remarkable algebraic facts almost immediately. Firsly, we have the following:

Lemma 4.3: *$T_{F_0}T_{F_1}(Z)$ is the zero object.*

Proof. Recall that by definition we have the exact triangles



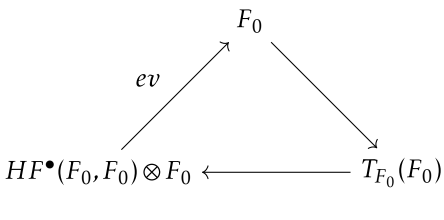
However, $HF^{\bullet}(F_1, Z) = \langle c \rangle$ is one-dimensional, where c is the intersection point. So the map ev is just c and the tensor product does nothing, showing that (since cones are unique up to isomorphism)

$$T_{F_1}(Z) \simeq T'_Z(F_1)[1] \simeq \tau_Z^{-1}(F_1)[1] \simeq F_0[1]$$

where in the last part we used 4.2. Now, we can apply the algebraic twist with F_0 and get:

$$T_{F_0}T_{F_1}(Z) = T_{F_0}(F_0[1]) = T_{F_0}(F_0)[1]$$

However, this twist comes from the exact triangle



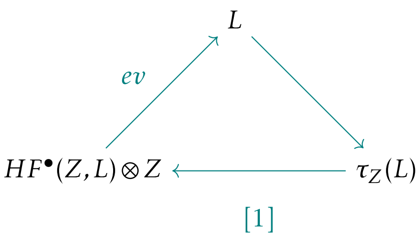
But by 2.8 the self Floer cohomology of $F_0 \simeq \mathbb{R}^n$ is just a copy of $\mathbb{C}$ and the evaluation map becomes the identity. We can conclude the proof by the general fact about triangulated categories which states that the cone of an isomorphism is zero. $\square$

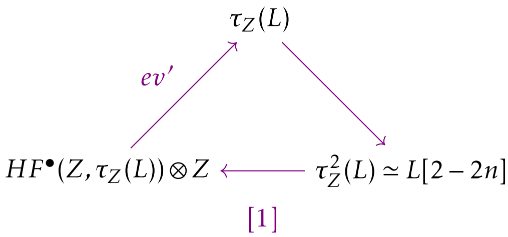
We are now in a position to show that any Lagrangian satisfying our original conditions can be achieved algebraically using only the Lagrangians F_0 and F_1, giving meaning to the phrase that they generate the Fukaya category.

Since L agrees with F_0 almost everywhere, the closure of $L \setminus F_0$ is a compact set which can be shrinked to lie arbitrarily close to the zero section, where it looks like the graph of an exact 1-form. In such a case, L is Lagrangian isotopic to $\tau_Z^2(L)$. However, the grading on the latter will have been changed by the Dehn twist, more precisely (this is shown in Seidel [20], Lemma 5.7)

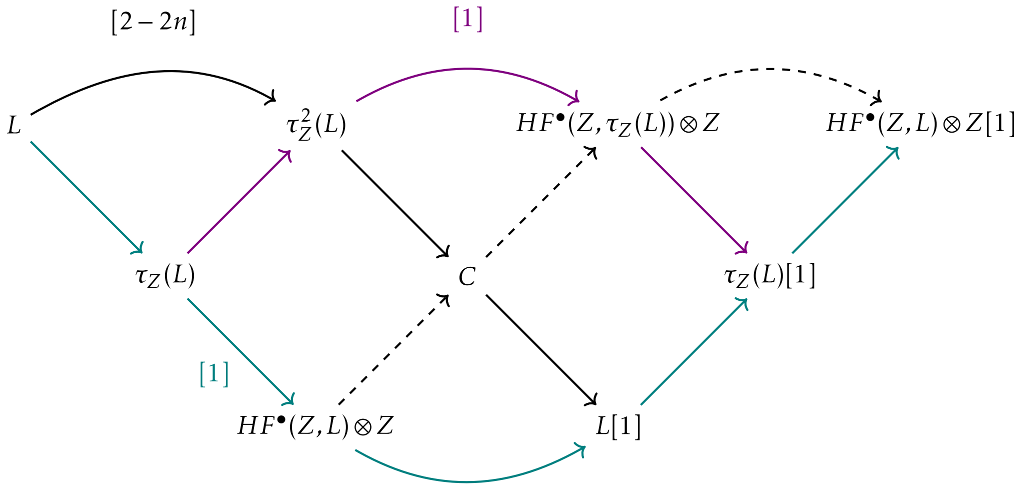
$$\tau_Z^2(L) \sim L[2-2n]$$

Now we can exploit this fact together with the octahedral axiom applied to the following two exact triangles:





The octahedral axiom allows us to "braid" these together as follows:



In other words, the cone in the middle is part of two distinguished triples, the two diagonal braids (the black one and the dashed one). Hence, we have that $L \to \tau_Z^2(L) \to C$ is a distinguished triangle and moreover there are dashed arrows fitting C into the other distinguished triangle

$$C \simeq \operatorname{cone}(HF^\bullet(Z, \tau_Z(L)) \otimes Z[-1] \to HF^\bullet(Z, L) \otimes Z)$$

However, when $n \geq 2$, the map $L \to \tau_Z^2(L) \simeq L[2-2n]$ is going into negative degrees, so is an element of $HF^{2-2n}(L, L) = 0$. By the fact that the cone of the zero map is a direct sum in a triangulated category, we get

$$C \simeq L \oplus L[2-2n-1]$$

However, we can also simplify the other distinguished triangle as well. In particular

$$HF^{\bullet}(Z, \tau_Z(L)) \simeq HF^{\bullet}(\tau_Z^{-1}(Z), L) = HF^{\bullet}(Z[n-1], L) \simeq HF^{\bullet+1-n}(Z, L)$$

The first isomorphism follows since Floer cohomology is unchanged by applying the Dehn twist, a symplectic isomorphism, to both Lagrangians, and moreover $\tau_Z(Z) \sim Z[1-n]$ since it doesn't change Z as a space but shifts the grading, by the same reasoning as before. The last isomorphism follows from the fact that

$$HF^{\bullet}(L_0[k], L_1[l]) \simeq HF^{\bullet-k+l}(L_0, L_1)$$

All of these facts are shown in Seidel, [20], page 12. We conclude:

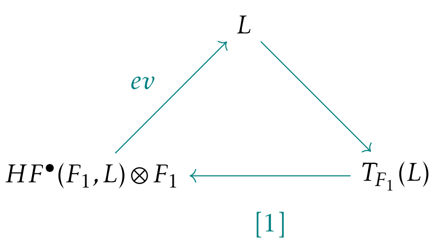
Lemma 4.4: *We have that*

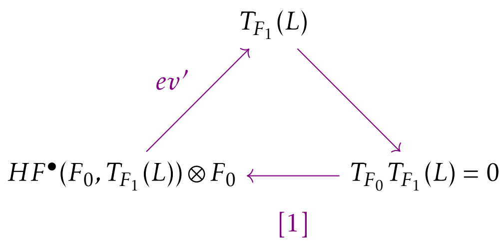
$$L \oplus L[1-2n] \simeq C \simeq \text{cone}(HF^{\bullet}(Z, L)[-n] \otimes Z \to HF^{\bullet}(Z, L) \otimes Z)$$

Moreover, $T_{F_0} T_{F_1}(L)$ is the zero object, just as we proved before for Z.

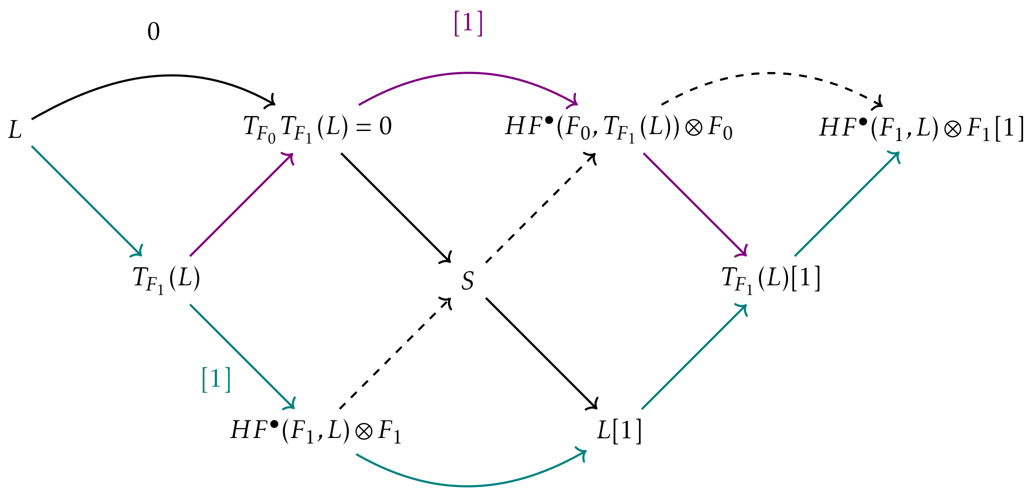
Proof. The first part is the calculation we just did. For the second part, notice that $T_{F_0} T_{F_1}$ respects sums and cones and hence sends the right hand side to 0, by 4.3. □

Let us now calculate L entirely in terms of F_0 and F_1. We can do the same trick as before, using the octahedral axiom, applied to the two exact triangles:





Braiding them gives us:



Now, we use the fact that the cone of the zero morphism is a direct sum:

$$S \simeq L[1] \oplus 0 = L[1]$$

On the other hand, this is precisely equal to the cone in the dashed diagonal, hence we have shown the following:

Proposition 4.5:

$$L[1] \simeq \text{cone}(HF^\bullet(F_0, T_{F_1}(L)) \otimes F_0[-1] \to HF^\bullet(F_1, L) \otimes F_1)$$

We are in the following situation: we have represented L as the cone of a map

$$x : HF^\bullet(F_0, T_{F_1}(L))[-1] \otimes F_0 \to HF^\bullet(F_1, L) \otimes F_1$$

of degree 0, i.e. if we put $V = HF^\bullet(F_0, T_{F_1}(L))[-1], W = HF^\bullet(F_1, L)$, then $x \in \text{Hom}(V, W) \otimes HF^\bullet(F_0, F_1)$. We computed $HF^\bullet(F_0, F_1) \simeq H^\bullet(S^{n-1}, \mathbb{C})$ in 4.1, having a generator a of degree 0 and b of degree $n-1$. Hence, we can write $x = \alpha \otimes a + \beta \otimes b$, where $\deg(\alpha) = 0, \deg(\beta) = 1 - n = d$, giving us a representation of the following quiver, called the graded Kronecker quiver:

$$V \underset{\alpha}{\overset{\beta}{\rightrightarrows}} W$$

In fact, this representation must be indecomposable: if not, then we would be able to express L as a direct sum, contradicting the fact that L is connected: $HF^0(L, L) \simeq H^\bullet(L; \mathbb{C}) \simeq \mathbb{C}$.

Now, applying $\text{hom}(L, -)$ to $L[1] = \text{cone}(V \otimes F_0 \xrightarrow{x} W \otimes F_1) = \text{cone}(A \to B)$ we get that $\text{hom}(L, L) = HF^\bullet(L, L)$ is the cohomology of the complex End(cone) having the matrix form

$$\left(\begin{pmatrix} \text{hom}(A, A) & \text{hom}(B, A) \\ \text{hom}(A, B) & \text{hom}(B, B) \end{pmatrix}, d = \left[\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, - \right] \right)$$

In the differential d, we have assumed $d_A = d_B = 0$ by moving to a minimal model. Now, we calculate using Lemma 4.1:

$$\text{hom}(A, A) = \text{hom}(V \otimes F_0, V \otimes F_0) = \text{hom}(V, V) \otimes HF^\bullet(F_0) = \text{hom}(V, V)$$

$$\text{hom}(B, A) = \text{hom}(W, V) \otimes HF^\bullet(F_1, F_0) = 0$$

$$\text{hom}(A, B) = \text{hom}(V, W) \otimes HF^\bullet(F_0, F_1) = \text{hom}(V, W) \otimes H^\bullet(S^{n-1}; \mathbb{C})$$

$$\text{hom}(B, B) = \text{hom}(W, W) \otimes HF^\bullet(F_1) = \text{hom}(W, W)$$

Given an element $\begin{pmatrix} v & 0 \\ u & w \end{pmatrix} \in \text{End(cone)}$, where $v \in \text{hom}(V, V), u \in \text{hom}(V, W) \otimes H^\bullet(S^{n-1}; \mathbb{C}), w \in \text{hom}(W, W)$ we see that

$$d \begin{pmatrix} v & 0 \\ u & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ u & w \end{pmatrix} \pm \begin{pmatrix} v & 0 \\ u & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x \circ v \pm w \circ x & 0 \end{pmatrix}$$

But $x \circ v = \begin{pmatrix} \alpha \circ v \\ \beta \circ v \end{pmatrix}, w \circ x = \begin{pmatrix} w \circ \alpha \\ w \circ \beta \end{pmatrix}$, so we can describe this as the cohomology of the two-step complex

$$0 \to \hom(V,V) \oplus \hom(W,W) \xrightarrow{(x \circ \circ x)} \hom(V,W) \otimes H^\bullet(S^{n-1};\mathbb{C}) \to 0$$

or better yet, since β has degree $1-n$:

$$0 \to \hom(V,V) \oplus \hom(W,W) \xrightarrow{\begin{pmatrix} -\alpha\circ & \circ\alpha \\ -\beta\circ & \circ\beta \end{pmatrix}} \hom(V,W) \oplus \hom(V,W)[1-n] \to 0$$

To finish up, we need to quote a result from the representation theory of quivers, which classifies all indecomposable representations of the graded Kronecker quiver, as found in Proposition 4 in [22]. There are essentially three cases, which have the following consequences on the cohomology of the two-step complex:

- $\mathcal{O}_X(k), k \in \mathbb{Z}$, in which case the cohomology is one-dimensional and concentrated in degree zero
- $\mathcal{O}_X/\mathcal{J}^k_{X,0}, k > 0$, in which case it is $2k$-dimensional and has generators in degrees $0, -d, ..., (1-k)d$ and $d+1, 2d+1, ..., kd+1$
- $\mathcal{O}_X/\mathcal{J}^k_{X,\infty}, k > 0$, in which case it is $2k$-dimensional with generators in degrees $0, d, ..., (k-1)d$ and $1-d, 1-2d, ..., 1-kd$

Corollary 4.6: *L is isomorphic, up to a shift, to* $\mathrm{cone}(F_0 \to F_1)$ *in* $D^b\mathrm{Fuk}(T^*S^n)$ *Moreover,* $H^\bullet(L) \simeq H^\bullet(S^n;\mathbb{C})$, $[L] = \pm[S^n]$ *and for any other L' satisfying the same conditions,* $L \cap L' \neq \emptyset$

Proof. Since $d = 1-n < 0$, none of the representations above can give us the cohomology of an n-dimensional manifold, except for the case $k=1$ in $\mathcal{O}_X/\mathcal{J}^k_{X,\infty}$, which is given by $\dim V = \dim W = k = 1$, and hence $V \otimes F_0 = F_0, W \otimes F_1 = F_1$, implying that $L = \mathrm{cone}(F_0 \to F_1)$. For this representation $\mathcal{O}_X/\mathcal{J}^1_{X,\infty}$, the cohomology of the two-step complex has generators in degrees $0, n$ as already mentioned, which shows that $H^\bullet(L) \simeq H^\bullet(S^n;\mathbb{C})$.

Now, F_0, F_1 and L fit into an exact triangle, giving a long exact sequence after applying $\hom(F_1, -)$. But by 4.1, we see that this implies $HF^\bullet(F_1, L) \simeq HF^\bullet(F_1, F_1) \simeq \mathbb{C}$, since $HF^\bullet(F_1, F_0) = 0$. A property of Floer cohomology is that the Euler characteristic calculates the intersection number: $\chi(HF^\bullet(F_1, L)) = F_1 \cdot L = \pm 1$, which implies the second part. For the last part, if L' satisfied the same conditions, then it would be isomorphic to L in the Fukaya category, and hence $HF^\bullet(L, L') \simeq HF^\bullet(L, L) \simeq H^\bullet(L;\mathbb{C}) \simeq H^\bullet(S^n;\mathbb{C}) \neq 0$ and hence they cannot be disjoint, as Floer cohomology of disjoint Lagrangians is zero. □

5 Conclusion

In the course of this essay, we developed the basic theory of symplectic geometry and Floer cohomology (through an analogy with Morse theory) and described the basic principles of the Fukaya category. Moreover, we showed how the Dehn twist, a symplectic automorphism, fits into an exact triangle in the derived Fukaya category. Strikingly, through purely algebraic manipulation, this allowed us to compute the Fukaya category of $T^*S^n, n > 1$, which suggests that, Floer-theoretically, every exact Lagrangian should behave exactly like the zero section.

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