QUANTUM COHOMOLOGY NOTES

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1. Introduction

We restrict ourselves today to only considering hypersurfaces in projective space, as it will be a nice, simple illustration of the general principles of GW theory.

2. Gromov-Witten invariants

We begin by introducing our first player: the projective space $\mathbb{P}^r$. Given some $\beta \in H_2(\mathbb{P}^r)$, say $d[L]$, we can consider the moduli space of genus zero curves with n marked points, which is formally the space of stable maps $f : C \to \mathbb{P}^r, f_*[C] = dL$. This is usually denoted by $\overline{M}_{0,n}(\mathbb{P}^r, dL)$.

Proposition 2.1. *The compactified moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, dL)$ is a smooth orbifold of the expected dimension.*

Proof. This is just a sketch, but we note the main important observation: for a moduli space to be smooth, what is reuired is that there be no obstructions to deforming maps. Suppose we have $f : C \to \mathbb{P}^r$. Deforming f is the same as deforming the graph $\Gamma_f \subset C \times \mathbb{P}^r$. The obstruction to doing this live in $H^1(\Gamma_f, \mathcal{N}_{\Gamma_f})$. Let's compute this by first considering the long exact sequence of

$$0 \to T_{\Gamma_f} \to T_{C\times\mathbb{P}^r} \to \mathcal{N}_{\Gamma_f} \to 0$$

This gives us that $H^1(\Gamma_f, \mathcal{N}_{\Gamma_f}) \simeq H^1(C, f^*T_{\mathbb{P}^r})$. Now consider the pullback of the Euler sequence:

$$0 \to f^*\mathcal{O}_{\mathbb{P}^r} \to f^*\mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r+1} \to f^*T_{\mathbb{P}^r} \to 0$$

Since $f_*[C] = dL$, that means that $f^*\mathcal{O}(1) = \mathcal{O}(d)$ and hence $H^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(1)) = 0$. This allows us to conclude that there are no obstruction to deforming! □

The words 'Deligne-Mumford stack' is a fancy way of saying some points in our moduli space have automorphisms. E.g. the map $f \in \mathcal{M}_{0,0}(\mathbb{P}^2, 2[L])$ given by $f([t_0 : t_1]) = [0 : t_0^2 : t_1^2]$ has an automorphism of order 2 given by $t_1 \mapsto -t_1$.

We note that the smoothness of the moduli space is a rare phenomenon - it has to do with the fact that $\mathbb{P}^r$ is a so-called *convex* variety: for any map $f : \mathbb{P}^1 \to \mathbb{P}^r$ of degree > 0, we have that $H^1(C, f^*T) = 0$. This is what allowed us to conclude there are no obstructions to deforming. Other examples of convex spaces are homogenous spaces. Moreover, even for $\mathbb{P}^r$ these moduli spaces are no longer smooth once we consider higher genus curves.

Example 2.2. Take $Bl_p\mathbb{P}^2$. The moduli space of degree 3 genus zero curves has a component consisting of cubics in $\mathbb{P}^2$ with at least one node, but not passing through p. There is also another component given by nodal curves $C_1 \cup C_2$ where C_2 is a double cover of the exceptional curve and C_1 is the strict transform of a cubic with a node on p.

Our solution to this problem today is to consider only hypersurfaces of degree m in projective space. Then, we can understand the moduli space of curves in $X \subset \mathbb{P}^r$ by thinking about the inclusion $\mathcal{M}(X,\beta) \to \mathcal{M}(\mathbb{P}^r, \iota_*\beta)$.

Proposition 2.3. *The moduli space of curves in X is cut out by a section of a vector bundle $\mathcal{E}$ on the moduli space of curves in $\mathbb{P}^r$. We can associate a virtual fundamental cycle with the following property:*

$$\iota_*[\overline{\mathcal{M}}_{0,n}(X, dL)]^{vir} = e(\mathcal{E}) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, dL)]$$

The vector bundle is given, over each $f : C \to \mathbb{P}^r$, as the cohomology group $H^0(C, f^*\mathcal{O}(m))$ and it carries a section $\tilde{s}$ induced by the section s cutting out X. For a generic section s, the zero locus of $\tilde{s}$ has the correct 'expected' dimension, i.e. it is transverse to the zero section. But for some s, even if X is smooth, this can fail!

Example 2.4. For example, take $X \subset \mathbb{P}^r$ cut out by a degree m equation. Then the moduli space of degree one curves is just the subset of the Grassmanian

$$\mathcal{M}_{0,0}(X, L) \subset \mathcal{M}_{0,0}(\mathbb{P}^r, L) \simeq \mathrm{Gr}(2, r+1)$$

cut out by the vector bundle $\mathrm{Sym}^l \mathcal{S}^*$, where $\mathcal{S}$ is the tautological bundle.

Now take quintic threefold. Generically, $\mathcal{M}_{0,0}(X, 1)$ is of expected dimension 0 and intersection theory computations show that there are 2875 lines. However, for the Fermat quintic, the moduli space of lines is given by the union of 50 plane quintics, so has dimension 1. Moreover, it is non-reduced. Instead, we somehow need to produce a zero-dimensional cycle inside of this 1-dimensional moduli space.

One solution to this problem is to use the normal cone construction. We are in the following situation: we have $\mathcal{M}$ the moduli space of curves in $\mathbb{P}^r$ and a vector bundle on it $\mathcal{E}$ cutting out $\mathcal{M}' = (V(\tilde{s}))$, the moduli space of curves on X.

Definition 2.5. The virtual fundamental class for a degree m hypersurface X in projective space is defined by

$$[\mathcal{M}(X,\beta)]^{vir} := 0^!_{\mathcal{E}}[\mathcal{M}(\mathbb{P}^r, dL)]$$

where the map $0^!_{\mathcal{E}} : A_\bullet(\mathcal{M}(\mathbb{P}^r, dL)) \to A_{\bullet - rk\mathcal{E}}(\mathbb{V}(\tilde{s}))$ is the virtual Gysin map defined by Fulton as the composition

$$A_\bullet(\mathcal{M}) \to A_\bullet(C_{\mathcal{M}'}) \to A_\bullet(\mathcal{E}|_{\mathcal{M}'}) \to A_{\bullet - rk\mathcal{E}}(\mathcal{M}')$$

This produces a class of the correct dimension. For a computation of the virtual class in the case of the Fermat quintic and a proof that one can still get a count of 2875 lines, see Katz-Albano and Clemens-Kley.

Now we are finally ready to define Gromov-Witten invariants.

Definition 2.6. The Gromov-Witten invariant is defined by

$$\langle \gamma_1, \dots, \gamma_n \rangle_{dL} := \deg \left((\mathrm{ev}_1^* \gamma_1 \cup \dots \cup \mathrm{ev}_n^* \gamma_n) \cap [\mathcal{M}_{0,n}(X, dL)]^{vir} \right)$$

2.1. **Axioms for Gromov-Witten invariants.** The Gromov-Witten invariants satisfy a set of recursive formulas, some of which we now describe.

Divisor axiom If $\beta \neq 0$ and $\gamma \in H^2$, then

$$\langle \gamma, \gamma_1, \dots, \gamma_n \rangle_\beta = \langle \gamma_1, \dots, \gamma_n \rangle_\beta \int_\beta \gamma$$

WDVV We sum over all splittings of a class $\beta = \beta_1 + \beta_2$ and over a basis and its dual. Fixing 4 classes $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ there is an equality

$$\sum \langle \gamma_1, \gamma_2, \phi_\alpha \rangle_{\beta_1} \langle \phi^\alpha, \gamma_3, \gamma_4 \rangle_{\beta_2} = \sum \langle \gamma_1, \gamma_4, \phi_\alpha \rangle_{\beta_1} \langle \phi^\alpha, \gamma_2, \gamma_3 \rangle_{\beta_2}$$

We note that there is an extension of WDVV to more insertions and also containing descendant invariants. There are more axioms, all of which can be found in Tseng's article, section 4.

The WDVV equation is particularly important, as it will lead to the associativity of the quantum product $(\gamma_1 * \gamma_2) * (\gamma_3 * \gamma_4) = (\gamma_1 * \gamma_4) * (\gamma_2 * \gamma_3)$.

IMPORTANT: there is also a notion of Gromov-Witten class, given by pushing down $\mathcal{M}_{0,n}(X, \beta)$ to X^n via the evaluation maps. The invariants can be extracted by intersecting with cycles in X^n. If the moduli space is algebraic, then all of this respects the Hodge decompositions. This will be useful when we want to say that the quantum product respects the Hodge decomposition. In other words, these Gromov-Witten classes are algebraic, so of type (p, p) and thus $\int_I a \otimes b \otimes c$ can be nonzero precisely when the Hodge degrees of a, b, c add up to a (p, p) class.

3. Quantum cohomology, big and small

Now that we have a readily available way to 'count' curves, we can construct a deformation of the cohomology ring of a variety.

Begin with the cohomology ring $H^\bullet(X)$, which has a basis $\phi_1, \phi_2, \ldots, \phi_m$. Then, we can define the small quantum product as follows:

$$\phi *_s \phi' = \sum \langle \phi, \phi', \phi_\alpha \rangle_\beta \phi^\alpha \mathbf{q}^\beta$$

where ϕ^α is the dual basis in cohomology to ϕ_α. Here, the $\mathbf{q}^\beta$ is a quantum variable with degree $c_1(\beta)$ which is encoding the fact that there is an invariant in the class of β. Moreover, the β run over the effective curve classes in $H_2(X)$.

The big quantum product allows arbitrarily many more insertions: it is formally written as

$$\phi *_t \phi' := \sum \langle \phi, \phi', \phi_\alpha, t, t, \ldots, t \rangle_\beta \phi^\alpha \mathbf{q}^\beta t^s$$

The parameter t here denotes a general element of cohomology $t = t_0\phi_0 + \cdots + t_m\phi_m$ and the t_i are coordinates on $H^\bullet(X)$. As such, for every $t \in H^\bullet(X)$ we have a different product.

This is a bit confusing, so let's dwell on it for a moment. We have $H^\bullet(X)$ which has elements of the form ϕ_α. But we are now thinking of it as an algebra over $\mathbb{C}[q, t]$ where the q, t also correspond to classes in $H^\bullet(X)$! We are allowed to specialize at specific points τ i.e. to set $t = sth, q = sth$, but we are not allowed to do the same for the ϕ's, since we are thinking of the big quantum cohomology as a family whose fiber as a vector space is $H^\bullet(X)$ and which over each point in $\mathrm{Spec}\mathbb{C}[q, t]$ gives a specific quantum product. The formal notion of things like these is called a Frobenius manifold, but that formalism is beyond the scope of this talk.

An elegant way to express the quantum product is via the so-called Gromov-Witten potential. Let t_α be the coordinate on $H^\bullet(X)$ corresponding to ϕ_α. Then the Gromov-Witten potential is a generating function of all Gromov-Witten invariants:

$$\Phi := \sum \langle \phi_0^{(n_0)}, \ldots, \phi_m^{(n_m)} \rangle_\beta \mathbf{q}^\beta \frac{t_0^{n_0}}{n_0!} \cdots \frac{t_m^{n_m}}{n_m!} = \sum \langle t, t, t, \ldots, t \rangle \mathbf{q}^\beta \frac{t^n}{n!}$$

This can be thought of as a function of $t \in H^\bullet(X)$ taking values in the Novikov ring. When we write $\phi_i^{(n_i)}$ we mean that there are n_i insertions in the invarant which are ϕ_i.

Theorem 3.1. *The big quantum product is associative. Equivalently, the Gromov-Witten potential satisfies the WDVV equations.*

Proof. We only sketch the main idea for the small quantum product, the general case being similar. The point is that the moduli space $\mathcal{M}_{0,n+4}(X,\beta)$ admits a forgetful map to $\overline{\mathcal{M}}_{0,4} \simeq \mathbb{P}^1$. On the moduli space of curves with 4 marked points, there are distinguished divisors $D(1,2|3,4)$, $D(1,4|2,3)$ and $D(2,4|1,3)$ which are linearly equivalent - they correspond to a stable degeneration when the fourth marked point joins one of the other three points which correspond to $0,1,\infty$. Then, we compute that

$$\int_{ft^*D(1,2|3,4)} ev_1^*\gamma_1 \cup ev_2^*\gamma_2 \cup ev_3^*\gamma_3 \cup ev_4^*\gamma_4 = \sum \langle \gamma_1,\gamma_2,\phi_\alpha\rangle_{\beta_1}\langle \phi^\alpha,\gamma_3,\gamma_4\rangle_{\beta_2}$$

To do this, we need to use the so-called cutting edge axiom. Firstly, the pullback of the divisor $D(1,2|3,4)$ is given by the nodal curves $C\cup C'$ (each of C, C' might also be nodal) such that the first one has marked points going to γ_1,γ_2 and the second one has marked points going to γ_3,γ_4. This can be expressed as a gluing of two moduli spaces along the diagonal, and the cutting edge axiom says that such moduli spaces satisfy

$$[\overline{\mathcal{M}}_{0,n_1+1}(X,\beta_1)\times_X \overline{\mathcal{M}}_{0,n_1+1}(X,\beta_1)]^{vir} = \Delta^![\overline{\mathcal{M}}_{0,n_1+1}(X,\beta_1)]^{vir}\boxtimes[\overline{\mathcal{M}}_{0,n_1+1}(X,\beta_1)]^{vir}$$

as in the diagram

$$\begin{array}{ccc} \mathcal{M}_{0,n_1+1}(X,\beta_1)\times_X \mathcal{M}_{0,n_2+1}(X,\beta_2) & \longrightarrow & \mathcal{M}_{0,n_1+1}(X,\beta_1)\times \mathcal{M}_{0,n_2+1}(X,\beta_2) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X\times X \end{array}$$

Finally, we need to use a formula for the diagonal class $\Delta = \sum \phi_\alpha\otimes\phi^\alpha$ and some intersection theory. □

3.1. **Kontsevich recursion.** We now apply the WDVV equations to compute the number of curves N_d of degree d in $\mathbb{P}^2$. Let us choose a basis ϕ_0,ϕ_1,ϕ_2 of cohomology and its dual coordinates t_0,t_1,t_2 so that $\phi_2\in H^4$ is Poincare dual to the fundamental class, i.e. it corresponds to a point insertion, ϕ_1 is the hyperplane and ϕ_0 is the unit in cohomology.

Because of the divisor axiom and the fact that ϕ_0 is a unit in big quantum cohomology (I have not proved this), the only interesting invariants on $\mathbb{P}^2$ are of the form

$$\langle \phi_2,\phi_2,\dots,\phi_2\rangle_d$$

The only time this invariant can be nonzero is when the virtual dimension is zero, so

$$\dim\mathbb{P}^2 + m - 3 + c_1(d) - m|\phi_2| = 3d-1-m = 0 \implies m = 3d-1$$

Let us call these invariants N_d. We can write the potential as follows:

$$\Phi = \frac{1}{2}(t_0t_1^2+t_0^2t_2) + \sum_d N_d e^{dt_1}\frac{t_2^{3d-1}}{(3d-1)!}$$

Now let's do WDVV for the quadruple $\phi_2,\phi_2,\phi_1,\phi_1$ splitting it into $\{\phi_2,\phi_2\}\cup\{\phi_1,\phi_1\}$ versus $\{\phi_2,\phi_1\}\cup\{\phi_2,\phi_1\}$. The resulting WDVV equation is

$$\frac{\partial^2}{\partial_{t_2}\partial_{t_1}}\partial_{t_1}\Phi\times\partial_{t_1}\frac{\partial^2}{\partial_{t_2}\partial_{t_1}}\Phi = \frac{\partial^2}{\partial_{t_2}\partial_{t_2}}\partial_{t_2}\Phi\times\partial_{t_0}\frac{\partial^2}{\partial_{t_1}\partial_{t_1}}\Phi + \frac{\partial^2}{\partial_{t_2}\partial_{t_2}}\partial_{t_1}\Phi\times\partial_{t_1}\frac{\partial^2}{\partial_{t_1}\partial_{t_1}}\Phi$$

We will write this as

$$\Phi_{112}^2 = \Phi_{222} + \Phi_{122}\Phi_{111}$$

After equating the terms in fromt of $e^{dt_1}t_2^{3d-4}/(3d-4)!$ one arrives at the identity

$$N_d = \sum_{d_1+d_2=d} N_{d_1}N_{d_2}\left(d_1^2d_2^2\binom{3d-4}{3d_1-2} - d_1^3d_2\binom{3d-4}{3d_1-1}\right)$$

3.2. **Example: the cubic fourfold, the classical way.** We will compute the ambient quantum cohomology of the cubic fourfold, following a classical approach by Beauville which works for low-degree complete intersections in projective space.

Let's consider $X_4 \subset \mathbb{P}^5$ and focus on the hyperplane-generated cohomology, i.e. the one inherited from the ambient projective space. First of all, $|\mathbf{q}| = c_1(\mathcal{O}(1)) = 3$ in the Chow ring, equivalently $2 \times 3 = 6$ in cohomology.

Firstly, 1 is a unit in quantum cohomology, so

$$h * 1 = h$$

Secondly, for degree reasons, since $h * h$ has degree 2 but q has degree 3 we have that

$$h * h = h^2$$

i.e. there is only a classical contribution. Now, we finally get something interesting: there is a potential contribution in

$$h * h^2 = h^3 + a\mathbf{q}$$

We need to compute this a which is given by the Gromov-Witten count

$$a = \langle h, h^2, \frac{1}{3}h^4 \rangle_1$$

since the dual to the pt is the fundamental class $h^4/3$.

But by the divisor axiom, we can take out the h and get

$$a = \frac{1}{3}\langle h^2, h^4 \rangle_1 = \int_{\mathcal{M}_{0,2}(X,1)} \mathrm{ev}_1^* h^2 \cup \mathrm{ev}_2^* h^4$$

But the variety of lines is a very classical object: firsly, on $\mathbb{P}^5$ it just comprises of the Grassmanian $\mathbb{G}(1,5) = \mathrm{Gr}(2,6)$. Over this Grassmanian there is the tautological 2-plane and 4-plane bundles $\mathcal{S}, \mathcal{Q}$ and the Fano variety of lines on a cubic X is cut out by a section of the vector bundle $\mathrm{Sym}^3\mathcal{S}^*$. So we can reduce the integral to an integral on the Grassmanian!

$$\langle h^2, h^4 \rangle_1 = \frac{1}{3}\int_{\mathrm{Gr}(2,6)} e(\mathrm{Sym}^3\mathcal{S}^*) \cup c_1(\mathcal{Q}) \cup c_3(\mathcal{Q})$$

The Euler class is telling us that we are intersecting with the class of the Fano variety of lines $[F_X]$, whereas the Chern classes of $\mathbb{Q}$ correspond to the h^2, h^4 insertions. One could either compute this directly, or plug in the following code into Macaulay2 online https://www.unimelb-macaulay2.cloud.edu.au/#home

```
loadPackage "Schubert2"
G = flagBundle({2,4})
(S,Q)=G.Bundles
V=symmetricPower_3 dual S
c=chern_4(V)*chern_1(Q)*chern_3(Q)
1/3*integral c
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The answer is $a = 6$, so $h * h^2 = h^3 + 6q$. Continuing, we need to compute

$$h * h^3 = h^4 + b\mathbf{q}h$$

but by the same logic, by usuing Macaulay again,

$$b = \frac{1}{3}\langle h^3, h^3 \rangle_1 = \frac{1}{3}\int_{\mathrm{Gr}(2,6)} e(\mathrm{Sym}^3\mathcal{S}^*) \cup c_2(\mathcal{Q}) \cup c_2(\mathcal{Q}) = 15$$

Finally, what remains is $h * h^4 = 0 + c\mathbf{q}h^2$ and

$$c = \frac{1}{3}\langle h^4, h^2\rangle_1 = \int_{\mathrm{Gr}(2,6)} e(\mathrm{Sym}^3\mathcal{S}^*) \cup c_3(\mathcal{Q}) \cup c_1(\mathcal{Q}) = 6$$

due to the same exact invariant.

Now that we know what $h * h^i$ is we can compute

$$h^{*5} = h * h * (h * h^2) = h * h * (h^3 + 6\mathbf{q}) = h * (h * h^3 + 6\mathbf{q}h) = h * (h^4 + 15\mathbf{q}h + 6\mathbf{q}h) = \\ = h * (h^4 + 21\mathbf{q}h) = h^5 + 27\mathbf{q}h^2$$

We conclude:

$$\mathrm{QH}^{amb}(X) = \mathbb{C}[\mathbf{q}, h]/(h^{*5} - 27\mathbf{q}h^{*2})$$

where by *amb* we mean that we are considering the part of the quantum cohomology inherited from the ambient $\mathbb{P}^5$.

4. Givental's approach and the quantum differential equation

While the previous approach works nicely for some examples, there is a much more general and fundamental approach due to Givental, relying on the so-called mirror theorem. It identifies solutions to two a priori very different differential equations.

First, we need to review what the descendant classes are. On the moduli space $\overline{\mathcal{M}}_{0,n}(X,\beta)$ there are n universal line bundles given by, at a point $f : C \to X$, the cotangent line at the marked point p_i. Using these, we can define Gromov-Witten invariants with descendant insertions

$$\langle \tau_{d_1}\gamma_1, \dots, \tau_{d_n}\gamma_n\rangle_\beta = \int_{\mathcal{M}_{0,n}(X,\beta)} \mathrm{ev}^*\gamma \cup \prod c_1(\mathcal{L}_i)^{d_i}$$

The reason we introduced these classes is that they satisfy a rich recursive structure and moreover can be used to construct a solution to the so-called quantum differential equation, which are zeros of the operator on cohomology-valued vectors:

$$\nabla_{\partial_q} = \partial_q + u^{-1}q^{-1}H * (-)$$

The WDVV equations imply this is flat.

Givental's idea is to construct a fundamental solution of this equation by considering descendant Gromov-Witten invariants.

He shows that, if $\gamma = \sum t_j\phi_j$ is a general element in cohomology and c is the first Chern class of the universal line bundle on the first marked point, then the collection of cohomology-valued functions

$$s_\alpha := \phi_\alpha + \sum_k \hbar^{-(k+1)} \sum_{(n,\beta)\neq(0,0)} \frac{1}{\underline{n}!}\langle \frac{\phi_\alpha}{\hbar - c}, \phi_i, \gamma^{(n)}\rangle \phi^i$$

forms a fundamental solution to the quantum differential equation.

Moreover, these fit together into what is called the J-function:

$$J := \sum \langle s_\alpha, 1\rangle \phi^\alpha$$

By using localization techniques to compute integrals over the moduli spaces, he concludes:

Theorem 4.1. *The J-function for Fano complete intersections in projective space is equal to the I-function, which is given (after restriction to the space generated by the hyperplane class H, in the case of hypersurfaces) by the hypergeometric function*

$$I(t) := e^{tH/\hbar} \sum_{d=0} e^{dt} \frac{\prod_{a=1}^{dm}(mH + a\hbar)}{\prod_{b=1}^{d}(H + \hbar b)}$$

Moreover, under the change of coordinates $q = e^t, u = -\hbar$, the I function satisfies the Picard-Fuchs equation

$$(\hbar\partial_t)^r I = (-1)^{r-1-m} qm \prod_{a=1}^{m-1} u(mq\partial_q + a)$$

Example 4.2. We consider $\mathbb{P}^1$ and the small J-function, i.e. we throw out t_i corresponding to $H^i, i \neq 2$. Then it has the following nice formula:

$$J = e^{tH/\hbar}(1 + \sum_{\beta\neq 0} q^\beta \langle \frac{\phi_\alpha}{\hbar - c}, 1\rangle_\beta \phi^\alpha) = e^{tH/\hbar}(1 + \sum_d q^d \left(\langle \frac{H}{\hbar - c}, 1\rangle_d + \langle \frac{1}{\hbar - c}, 1\rangle_d H \right))$$

Now we use the following two descendant invariants, computed by Pandharipande:

$$\langle \tau_{2d-1} H, 1\rangle_d = \frac{1}{(d!)^2}, \langle \tau_{2d}, 1\rangle_d = \frac{-2}{(d!)^2}(1 + \frac{1}{2} + \cdots + \frac{1}{d})$$

to rewrite this as

$$e^{tH/\hbar}\left(1 + \sum_{d\geq 1} q^d (\frac{\hbar^{-2d}}{(d!)^2} + \hbar^{-(2d+1)} \frac{-2}{(d!)^2} H_d H)\right) =$$

$$= e^{tH/\hbar}(1 + \sum_{d\geq 1} \frac{q}{\hbar^2}^d \frac{1}{(d!)^2}(1 - 2\frac{H}{\hbar} H_d)) =$$

$$= e^{tH/\hbar} \sum q^d \frac{1}{(d!\hbar^{d-1} H_d H + d!\hbar^d)^2} =$$

$$= e^{tH/\hbar} \sum q^d \frac{1}{(H + \hbar)(H + 2\hbar)\dots(H + d\hbar)^2} = I_{\mathbb{P}^1}$$

Example 4.3. In this example, we use the mirror theorem to compute the action by $H*$ on the cubic fourfold. A solution of the quantum differential equation on a cubic fourfold would have to satisfy

$$uq\partial_q \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = - \begin{pmatrix} 0 & 0 & aq & 0 & 0 \\ 1 & 0 & 0 & bq & 0 \\ 0 & 1 & 0 & 0 & cq \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

where we pretend we don't know what a, b, c are. Writing $\psi_4 = \psi, D = uq\partial_q$, this implies that

$$\psi_3 = -uq\partial_q \psi = -D\psi$$
$$\psi_2 = -D\psi_3 = D^2\psi$$
$$D\psi_2 = -\psi_1 - cq\psi \implies \psi_1 = -cq\psi - D^3\psi$$
$$D\psi_1 = -\psi_0 - bq\psi_3 = -\psi_0 + bqD\psi$$
$$D\psi_0 = -aq\psi_2 = -aqD^2\psi$$

But

$$D\psi_1 = D(-cq\psi - D^3\psi) = -cqD\psi - ucq\psi - D^4\psi$$

so

$$\psi_0 = (bq + cq)D\psi + ucqD\psi + D^4\psi$$

Thus,

$$-aqD^2\psi = D\psi_0 = uq(b+c)D\psi + q(b+c)D^2\psi + u^2cq\psi + ucqD\psi + D^5\psi$$

i.e.

$$D^5\psi + q(D^2(b+c) + uD(b+2c) + u^2c)$$

On the other hand, Givental's mirror theorem relates this to the Picard-Fuchs equation

$$D^5\psi = -q(3(3D+u)(3D+2u))\phi$$

Matching the coefficients we recover $a = 6, b = 6, c = 15$ as before.

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