Mirror Symmetry for Hypersurfaces in Toric Varieties

Daniel Simms, Bogdan Simeonov

Contents

0 Introduction

Mirror symmetry is a duality phenomenon first observed by physicists in the 80s. In short, it posits that spaces X (originally Calabi-Yau threefolds used in models of string theory) may admit mirror spaces $X^\vee$ which give the same physical theory. This was first used to great effect in exhibiting how certain Gromov-Witten invariants on one side can be translated to period equations on the other and computed, for example in the case of the quintic threefold.

It took some time for these ideas to be dressed in mathematical language. There are now two complementary approaches to understanding mirror symmetry in mathematics: the geometric approach of Strominger-Yau-Zaslow (SYZ) and the homological approach of Kontsevich (HMS).

Roughly, the main idea behind SYZ is that mirror pairs come from dual Lagrangian torus fibrations. However, such torus fibrations $X \to B$ may be singular, hence the construction of the mirror is not always straightforward: it needs to be corrected by what are called *instanton corrections* along walls where the Lagrangian tori are singular and bound holomorphic disks, obstructing their Floer theory.

The SYZ approach also allows for a direct translation in the HMS picture from the symplectic A-side to the algebraic B-side. Concretely, given a Lagrangian $L \subset X$, one can first study its intersections with the torus fibers and calculate the Floer cohomology $HF(L,T)$. On the mirror, the torus fibers correspond to skyscraper sheaves, so the Floer cohomology should match the Ext group $\mathrm{Ext}(L^\vee, \mathcal{O}_p)$ and gives an understanding of the support of the sheaf (or complex of sheaves) mirror to L. For example, a Lagrangian section of the torus fibration in this way corresponds to a line bundle on the total space of the mirror.

In this note, we describe a general approach to SYZ given by Auroux-Abouzaid-Katzarkov [AurAb2015] that aims to develop mirror symmetry for hypersurfaces in toric varieties. Fix a toric variety V with dense open torus V^0, and a hypersurface $H \subset V$ cut out by a Laurent polynomial f. In short, while it is in general difficult to find a Lagrangian torus fibration on H, one instead replaces it with the *deformation to the normal cone* $X := \mathrm{Bl}_{H\times 0} V \times \mathbb{C}$ which inherits a torus fibration from V. There is an open subset X^0 of this blowup,

$$X^0 = \{xy = f(z)\}$$

which is a conic fibration whose singular fibers are precisely those lying over $H^0 = \mathbb{V}(f) \subset V^0$. In the paper [AurAb2015], a Lagrangian torus fibration is constructed from such a conic fibration which has a wall-chamber decomposition encoded by the geometry of the tropical limit of H. Once this is understood and appropriate instanton corrections are made, the Newton polytope of f is used to build a toric variety Y and a superpotenial W on it. The idea is that there will be an open set $Y^0 = Y \ W^{-1}(0)$ mirror to X^0, while (Y^0, W) will be mirror to X.

In fact, this recipe can be reversed: one can start from Y^0 which can be thought of as a degeneration of algebraic tori to the toric boundary of Y. The fibers have a natural T^{n-1} action inherited from the torus action on Y, and one can construct a torus fibration by taking circles on the base of $Y \to \mathbb{C}$ and lifting them by the T^{n-1} action. This construction is simpler, as it has a single wall crossing (crossing the singular toric boundary fiber), and the chambers can be glued to produce exactly the conic fibration X^0.

We will explore these ideas with a key example in mind: the affine quadric T^*S^2 and its hyperkahler rotation $K_{\mathbb{P}^1}$. We will show that they are almost mirror: either one or the other has to be equipped with a superpotential under the SYZ constriction.

This note is organized as follows:

- In the first section, we mention briefly generalities about SYZ. This includes the deformation theory of special Lagrangian submanifolds, the definition of the superpotential and the construction of coordinates on the moduli space of Lagrangian tori. We then mention the well-understood case of Fano toric manifolds, where there is no wall-crossing and the disk potential is explicitly computed, as in [ChoOh].
- In the second section, we describe two simple examples. The first is that of the binodal cubic: this example can be either thought of as a conic fibration, or as an open subset of the toric variety $\mathbb{C}^2$ and both constructions can be applied, exhibiting it as self-mirror. The second example is the canonical bundle of $\mathbb{P}^1$, which we show is mirror (after removing a generic fiber) to an open subset of the affine quadric.
- In the third section, we give a sketch of the AAK constructions and mention how the construction applies to $K_{\mathbb{P}^n}$
- In section 4, we prove homological mirror symmetry for T^*S^2, i.e. exhibit an equivalence

$$\mathcal{W}(T^*S^2) \simeq \mathrm{MF}(K_{\mathbb{P}^1} \setminus \mathbb{C}^\times, x^2 z)$$

1 Foundations of SYZ

We begin with a purely motivational homological mirror symmetry perspective, and then move on to understand the moduli spaces of Lagrangian tori.

1.1 HMS perspective

Suppose X is mirror to $X^\vee$. Given a point $p \in X^\vee$, we can compute the endomorphisms of the skyscraper sheaf $\mathcal{O}_p$ in the derived category:

$$\mathrm{Ext}(\mathcal{O}_p, \mathcal{O}_p) \simeq H^\bullet(T^n, \mathbb{C})$$

This is done via a Koszul resolution computation, which we include here for sake of completeness.

Remark 1.1 (Koszul resolutions): Let $Y \subset X$ be a subvariety, which we can Koszul resolve:

$$\ldots \to \mathcal{E}^\vee \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

i.e. we replace $\mathcal{O}_Y = \bigwedge^\bullet \mathcal{E}^\vee$. Then the sheaf Ext's $\mathcal{E}\mathrm{xt}(\mathcal{O}_Y, \mathcal{O}_Y)$ are computed by applying $\mathcal{H}om(-, \mathcal{O}_Y)$ to the complex and taking cohomology. But once we restrict to Y, the maps become zero since they are given by the defining function for Y. Hence, the complex is formal and we get

$$\mathcal{E}xt^i(\mathcal{O}_Y, \mathcal{O}_Y) = \mathcal{H}om(\bigwedge^i \mathcal{E}^\vee, \mathcal{O}_Y) = \bigwedge^i \mathcal{E} \otimes \mathcal{O}_Y = \bigwedge^i \mathcal{E}|_Y = \bigwedge^i \mathcal{N}_{Y/X}$$

This allows us to compute the self Ext groups of a point, by also using the local-to-global spectral sequence:

$$H^q(X, \mathcal{E}xt(\mathcal{O}_p, \mathcal{O}_p)) \implies \mathrm{Ext}^n(\mathcal{O}_p, \mathcal{O}_p)$$

The local homs will be supported at p. So it has no higher cohomology i.e. the cases $q > 0$ give nothing. But when $q = 0$, we have no differential. So in fact

$$\mathrm{Ext}^n(\mathcal{O}_p, \mathcal{O}_p) \simeq H^0(X, \mathcal{E}xt^n(\mathcal{O}_p, \mathcal{O}_p))$$

which is an exterior algebra.

HMS predicts that this should correspond to some Lagrangian L in the Fukaya category with $\mathrm{End}(L) = H^\bullet(T^n, \mathbb{C})$. In particular, this would mean that

$$\mathrm{End}(L) = HF^\bullet(L, L) \simeq H^\bullet(L)$$

and so L has the same cohomology as a torus. A reasonable guess would be that L should be a Lagrangian torus.

If we think of X as a moduli of its points i.e. skyscraper sheaves, then we can reduce the problem of finding a mirror to constructing a suitable moduli space of (special) Lagrangian tori. The SYZ philosophy is that this moduli space should be exhibited by finding a torus fibration: this is not so straightforward as it may seem, as these fibrations may be singular, leading to a phenomenon known as *wall-crossing*.

Example 1.2: Let's think about the simplest example, that of $\mathbb{C}^\times$ with a Lagrangian circle fibration using the map

$$\mathbb{C}^\times \xrightarrow{\log|-|} \mathbb{R}$$

The fibers of this are concentric circles around the origin. In this case, the moduli space is parametrized by the base $\mathbb{R}$. But $\mathbb{C}^\times$ is *self-mirror*, so we need to somehow complexify this moduli space. This is done by introducing local systems: we not only consider the circles, but also equip them with a $U(1)$ local system. They are in bijection with

$$\mathrm{Hom}(\pi_1(S^1), U(1)) \simeq U(1)$$

and hence the moduli space complexifies to $\mathbb{R} \times U(1) \simeq \mathbb{C}^\times$.

SYZ gives us a hands on, geometric description of mirror symmetry and in fact allows us to understand mirrors to Lagrangians, via what is often called an SYZ (or Legendre) transform. If a sheaf (or complex of sheaves) $\mathcal{E}$ is mirror to L then, assuming HMS, we would have that

$$HF^\bullet(L, T) \simeq \mathrm{Ext}(\mathcal{E}, \mathcal{O}_p)$$

So we can test the support of the mirror by calculating the Floer cohomology of the Lagrangian with respect to the torus fibers!

Example 1.3 (Continued): In the previous example, the concentric circles in $\mathbb{C}^*$ or equivalently the circles of a given height in the cylinder correspond to points in the mirror. To find the structure sheaf $\mathcal{O}$, which is the trivial line bundle, by the above, we must have

$$HF^\bullet(L, T) \simeq \mathrm{Ext}(\mathcal{O}, \mathcal{O}_p) \simeq \mathbb{C}, \text{ for all } p \in \mathbb{C}^*$$

Therefore, the Lagrangian L intersects all circles once, and the most natural choice is the the cotangent fiber F. One can even show that the cotangent fiber intersects its wrapped image infinitely many times (one for each integer) and that the Floer differential in wrapped Floer cohomology vanishes by index considerations, so the following identity holds true:

$$HW^\bullet(F, F) \simeq \mathrm{Ext}(\mathcal{O}, \mathcal{O}) \simeq H^0(\mathcal{O}) = \mathbb{C}[t^\pm]$$

1.2 Structure of the moduli space

Now we review some material on the moduli spaces of special Lagrangian tori. In this section we will stick to special Lagrangians, as their deformation theory is particularly nice and illuminates the integral lattice perspective. In the subsequent chapters, it will turn out too difficult to construct special Lagrangian fibrations and we will drop that assumption.

Deformations of special Lagrangians

Let us for the moment consider almost Calabi-Yau manifolds, i.e. complex manifolds X^0 equipped with a nonvanishing holomorphic volume form $\Omega \in H^0(X^0, \Omega^n_{X^0}) = \Omega^{n,0}(X^0)$. The example we will be interested in will be $X^0 = X \setminus D$, where D is an anticanonical divisor on an arbitrary complex manifold X. The point is that the mirror to X is obtained from the mirror to X^0 by adding in terms to the *superpotential*, so we ought to understand X^0 first.

The restriction of this to a Lagrangian is a nowhere vanishing complex n-form. The *argument* is the scalar difference between $\Omega|_L$ and the induced volume form from the Kahler metric.

> **Definition 1.4 (Special Lagrangian):** *A Lagrangian is called* special *if the argument is constant.*

Let us consider the deformation theory of such special Lagrangians, which will describe the tangent space to their moduli space (before adding in local systems). Concretely, a deformation is equivalent to a section $\nu \in H^0(L, \mathcal{N}_{L/X^0})$, which we can think of as a tangent vector on L.

For the deformation to stay Lagrangian, we would like

$$0 = \mathcal{L}_\nu \omega = d\iota_\nu \omega$$

by Cartan's magic formula. Similarly, for the deformation to be special Lagrangian, we would like

$$0 = \mathcal{L}_\nu \operatorname{im}\Omega = d\iota_\nu \operatorname{im}\Omega$$

Defining the one-form $\alpha = -\iota_\nu \omega$ and also the $n-1$ form $\beta = \iota_\nu \operatorname{im}\Omega$, one has the relationship

$$\beta = \psi *_g \alpha$$

where ψ is the ratio between $\Omega|_L$ and vol_g and $*$ is the Hodge star. We saw that the deformation is special Lagrangian iff both α and β are closed and hence deformations to L through special Lagrangians are parametrized by ψ-harmonic forms:

> **Theorem 1.5 (McLean):** *The deformations of a special Lagrangian are controlled by the ψ-harmonic forms, which by Hodge theory correspond to $H^1(L;\mathbb{R})$:*
>
> $$\{\textit{special Lagrangian deformations of } L\} \leftrightarrow \mathcal{H}^1_\psi(L) = \{d\alpha = d^*(\psi\alpha) = 0\} \simeq H^1(L;\mathbb{R})$$

Remark 1.6 (Dual tori): In some sense, this should be the tangent space of the base in the torus fibration. Importantly, this contains an integral lattice $H^1(L;\mathbb{Z})$. We could have considered the dual viewpoint (prioritizing β to α), which would have given us a correspondence with $H^{n-1}(L;\mathbb{R})$. These correspond to dual tori:

$$H^1(L;\mathbb{R})/H^1(L;\mathbb{Z}) \simeq (H^{n-1}(L;\mathbb{R})/H^{n-1}(L;\mathbb{Z}))^\vee$$

In fact, these are the two dual integral affine structures on the base.

Now, we add in the local systems to complexify the moduli space. We can represent a local system $\nabla = d + iA$, where we choose A to be a ψ-harmonic 1-form. Because the space of connections is affine, the tangent space to the space of local systems can then be identified with the ψ-harmonic 1-forms A.

Definition 1.7 (Complex structure on the moduli space): *The moduli of special Lagrangians equipped with local systems has the tangent space*

$$T_{L,\nabla}M = \{(\nu, A) \in H^0(L, \mathcal{N}_{L/X}) \times \Omega^1(L)\} \simeq \mathcal{H}^1_\psi(L) \otimes \mathbb{C}$$
$$(\nu, A) \mapsto iA - \iota_\nu \omega$$

We use the complex structure on the right hand side to equip the moduli space with a complex structure:

$$\begin{array}{ccc} (\nu, A) & \longleftrightarrow & -\iota_\nu\omega + iA \\ \downarrow & & \downarrow \\ (\nu', A') & \longleftrightarrow & -A - i\iota_\nu\omega \end{array}$$

where $A' = -\iota_\nu\omega$ and ν' the normal vector such that $\iota_{\nu'}\omega = A$. This comes with a holomorphic n-form $\Omega^\vee$ which integrates:

$$\Omega^\vee_{L,\nabla}(\nu_j, A_j) := \int_L \bigwedge_j iA_j - \iota_{\nu_j}\omega$$

The Kahler form on it is given by

$$\omega^\vee_{L,\nabla}((\nu_1, A_1), (\nu_2, A_2)) := \int_L A_2 \wedge \iota_{\nu_1} Im\Omega - A_1 \wedge \iota_{\nu_2} Im\Omega$$

There is a projection map $\mathcal{M} \to B$ which forgets the connection. Here, B is the moduli of special Lagrangians. The fibers of this are tori and they can be checked to be special Lagrangian.

Remark 1.8 (Exhanging complex and symplectic structures): The holomorphic n-form on the B-side mirror is defined via the symplectic form on the A-side, and conversely the symplectic form on the B-side is defined using the holomorphic n-form on the A-side. In this way, mirror symmetry exchanges the complex and symplectic structures.

Coordinates on the moduli space

Observation (Coordinates on the moduli space): *To get coordinates on B, first choose a reference point L_0 and recall that $T_{L_0}B \simeq H^1(L_0;\mathbb{R})$ by sending v to $-\iota_v\omega$. By choosing a basis of $H_1(L_0)$ given by a set of loops $\gamma_1,\dots,\gamma_n$, we get an isomorphism*

$$T_{L_0}B \simeq \mathbb{R}^n$$
$$v \mapsto \left(\int_{\gamma_i} -\iota_v\omega\right)_i$$

Now, if v deforms L_0 to L, then we can trace out a cylinder Γ_i which is the flow of the curve γ_i. We see that $\int_{\Gamma_i}\omega = \int_{\gamma_i}\iota_v\omega$ and so, by using the exponential map, we have coordinates in a neighbourhood of L given by $\exp(-\int_{\Gamma_i}\omega)$.

Now we aim to complexify this construction to get local coordinate charts on $\mathcal{M}$ by weighting using the local systems. In this way, given a reference fiber L, the coordinates of L' will be given by

$$(L,\nabla) \mapsto \left(\exp\left(-\int_{\Gamma_i}\omega\right)\mathrm{hol}_\nabla(\partial\gamma_i)\right)_i$$

To remove dependence on the reference fiber, we can 'cap off' the cylinders Γ_i and consider an arbitrary $\beta \in H_2(X,L)$, for example classes represented by a disk.

Definition 1.9 (Functions on the mirror): *Let us introduce, for any $\beta \in H_2(X,L)$, a function*

$$z_\beta : \mathcal{M} \to \mathbb{C}^\times$$
$$z_\beta(L,\nabla) = \exp(-\omega\cdot\beta)\mathrm{hol}_\nabla(\partial\beta)$$

If we arrange that the class β deforms with L, this map is holomorphic with respect to the complex structure introduced before.

Proof. We have

$$d\log z_\beta = d\left(-\int_\beta \omega\right) + d\log\mathrm{hol}_\nabla(\partial\beta)$$

Consider what happens to the first: we must evaluate at $(v,A) \in C^\infty(NL)\oplus\Omega^1(L)$. The normal vector field produces a flow Ψ which transports the Lagrangians L to a one-parameter family L_t. The first summand is then

$$-\frac{d}{dt}|_{t=0}\int_{\beta_t}\omega = -\frac{d}{dt}|_{t=0}\int_\beta (\Psi^t)^*\omega =$$
$$= -\int_\beta \mathcal{L}_v\omega = -\int_\beta d\iota_v\omega = -\int_{\partial\beta}\iota_v\omega = -\langle\iota_v\omega,\partial\beta\rangle$$

The second summand can be computed explicitly, since we are working with a unitary connection over the trivial line bundle on L. Then the monodromy equation $d + tA = 0$ is solved by a curve of the form $\exp(-At)$ and what we get is just A. In this way,

$$d\log\mathrm{hol}(\partial\beta)(v,\alpha) = i\int_{\partial\beta}\alpha = i\langle\alpha,\partial\beta\rangle$$

All in all, we get

$$d\log z_\beta : (\nu,\alpha) \mapsto \int_{\partial\beta} -\iota_\nu\omega + i\alpha$$

The complex structure is given by $J(\nu,\alpha) = (a, -\iota_\nu\omega)$ where $\iota_a\omega = \alpha$. We immediately see that $d\log z_\beta$ commutes with this, so z_β is holomorphic! □

Remark 1.10: By the above formula, the holomorphic volume form can be reinterpreted as $d\log z_1 \wedge \cdots \wedge d\log z_n$ by choosing a basis of $H_1(L)$.

Remark 1.11: This construction locally identifies B with $H^1(L,\mathbb{R})$. We could have used $\iota_v\Omega$ instead, which would have identified it with a domain in $H^{n-1}(L,\mathbb{R})$ instead, which is the dual integral affine structure.

Finally, to build the mirror, we need to glue different neighbourhoods of Lagrangian tori $U^\vee$ subject to some *instanton corrections*: the gluing will be governed by a special global function on the mirror called the *superpotential*.

Superpotential

We are ready to define the disk potential, which is a way to encode in a single function all of the Maslov 2 disks bounded by a given Lagrangian L. It looks as follows:

$$W_L := \mathfrak{m}_0(L,\nabla) := \sum_{\mu(\beta)=2} n_\beta(L)\exp\left(-\int_\beta \omega\right)\mathrm{hol}_\nabla(\partial\beta) : \mathcal{M} \to \mathbb{C}$$

Here, n_β counts holomorphic disks passing through some generic point of $p \in L$. Since the z_β are holomorphic and in ideal situations the counts $n_\beta(L)$ are locally constant, we see that it is a holomorphic function on $\mathcal{M}$.

The superpotential is a curvature term in Floer theory: the differential of the complex $CF((L,\nabla),(L',\nabla'))$ squares to $(W(L,\nabla) - W(L',\nabla'))id$ and hence the Floer cohomology is well-defined only locally.

Remark 1.12 (Wall-crossing corrections): It is possible that a disk β of Maslov index 2 can break into a Maslov 2 disk and a Maslov 0 disk when moving through the moduli space of Lagrangians, with the Maslov 0 disk being bounded by L. After this, the Maslov 0 disk vanishes ('bubbles off'). To account for this, we need to introduct corrections: the moduli space breaks up into *chambers* separated by *walls*, which consists of the Lagrangians bounding Maslov zero disks. In other words, there are chambers of unobstructed Lagrangians, which give a collection of open charts, which then need to be glued once one crosses a wall. Work of Fukaya-Oh-Ohta-Ono and Auroux-Abouzaid-Katzarkov [AurAb2015] suggest that the coordinate change should look like

$$z_\beta \mapsto z_\beta h(z_\alpha) \quad \text{for} \quad h(z_\alpha) = 1 + O(z_\alpha).$$

Here the class β deforms to $\beta + \alpha$ after crossing a wall, with α representing a bubble holomorphic disk of Maslov index 0 bounding a singular Lagrangian on the wall.

Remark 1.13 (Maslov index and intersection number): In fact, in $X^0 = X \setminus D$ where D is an anticanonical divisor, the Maslov index of a disk with boundary on L is actually given by twice its intersection number with D, which is proved in [ChoOh]:

$$\mu(\beta) = 2\beta \cdot D$$

Hence, the superpotential is a sum over disks which intersect the anticanonical divisor once.

1.3 Toric Fano case

Have X a toric variety with ample anticanonical bundle. As a toric variety it contains a compact torus $T^n = (S^1)^n$ acting on X as an extension of multiplication on $(\mathbb{C}^\times)^n \subset X$. This action has a moment map

$$\mu : X \to \mathfrak{g}^* \simeq \mathbb{R}^n$$

The image of μ is a Delzant polytope Δ. The interior has $\mu^{-1}(\Delta^\circ) \simeq (\mathbb{C}^\times)^n$ with the standard torus action. For any facet F of Δ we have $\mu^{-1}(F) = D_F$ a hypersurface and the toric invariant divisor is

$$\mu^{-1}(\partial\Delta) = \sum D_F = -K_X$$

as can be seen by looking at the canonical volume form

$$\Omega = \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}$$

on the torus which has poles precisely on the hypersurfaces.

Orbits under the torus action are special Lagrangians and we have a Lagrangian torus fibration

$$X \setminus D \to \Delta^\circ$$

.

We can put coordinates on the moduli space of Lagrangian tori as follows:

$$z_j(L,\nabla) = \exp(-2\pi\phi_j(L))\mathrm{hol}_\nabla(\gamma_j)$$

where $\phi_j = \pi_j \circ \mu$ and γ_j is the loop given by the j-th circle in T^n.

Theorem 1.14 (Cho-Oh): *The mirror to a toric variety X is $(\mathbb{C}^\times)^n$ equipped with the superpotential*

$$W = \sum_F e^{-2\pi\alpha(F)}\mathbf{z}^{v(F)}$$

with $v(F) \in \mathbb{Z}^n$ being the primitive integer normal to a facet F and $\alpha(F) \in \mathbb{R}$ the constant such that F has equation $\langle v(F), \phi\rangle + \alpha(F) = 0$.

The point is that all Lagrangians in $(\mathbb{C}^\times)^n = X \setminus D_F$ are unobstructed, so there is no wall crossing. Then, one identifies the Maslov 2 disks: for each facet, there is a unique class of disks contributing to the superpotential. In fact, strictly speaking, the mirror defined using the coordinates z_j is a subset of $(\mathbb{C}^\times)^n$ consisting of pairs (L,∇) where each term in the superpotential has norm less than 1. However, one can complete it by enlarging the moment polytope.

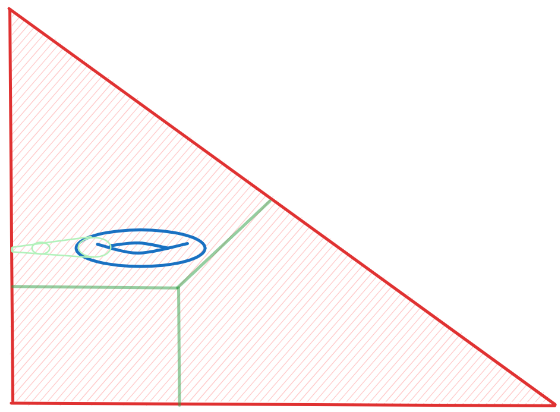


Figure 1: Standard product torus in $(\mathbb{C}^\times)^2$. The superpotential for it in $\mathbb{CP}^2$ is $W = z + z' + 1/zz'$

Remark 1.15 (Novikov coefficients): Formally, instead of exp, one should use a Novikov parameter T and consider mirrors as families, or equivalently as analytic spaces defined over the Novikov field. We will be a bit imprecise when it comes to this distinction.

2 First examples of wall crossing

In the last section, we saw that if we take a toric variety together with its toric boundary as the anticanonical divisor, essentially everything is known: the mirror moduli space is $(\mathbb{C}^\times)^n$ and the superpotential can be worked out from the facets of the toric polytope.

However, we could have chosen a different anticanonical divisor, which would greatly change the geometry of the moduli space. Instead of three lines in $\mathbb{C}P^2$, we could instead choose a line, together with a conic $xy = \epsilon$. Let us explore this example further.

2.1 The binodal cubic as a degeneration of $\mathbb{C}^\times$

We can think our example as $\mathbb{C}^2 = X$ relative to the divisor $D = \{xy = \epsilon\}$. We equip it with the holomorphic volume form $\frac{dx \wedge dy}{xy - \epsilon}$ and the projection to $\mathbb{C}$ given by xy. The picture is as follows:

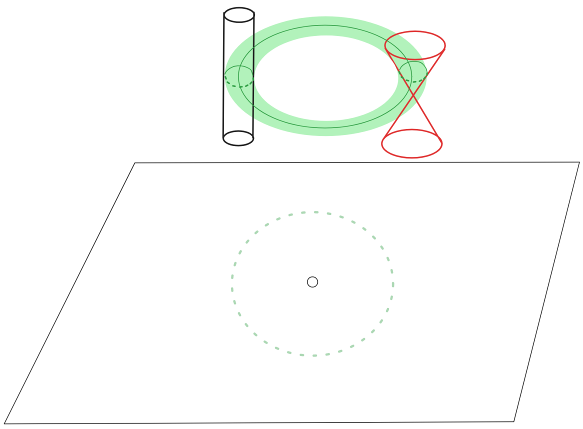


Figure 2: The binodal cubic $\mathbb{C}^2 \setminus D$ has a missing fibre in the middle, a generic fibre $\mathbb{C}^*$ and a single singular fibre consisting of a union $\mathbb{A}^1 \cup \mathbb{A}^1$

We will construct an SYZ mirror for $\mathbb{C}^2$ relative to this divisor, which will be the same geometrically as the mirror to $\mathbb{C}^2 \setminus D$, except that it will be equipped with a superpotential.

To begin with, we construct a singular Lagrangian torus fibration as follows: on the base, we take circles going around the missing point ϵ and then lift them to tori by choosing a height level, which is really a level set for the moment map of the S^1 action $e^{i\theta} \cdot (x, y) = (e^{i\theta} x, e^{-i\theta} y)$ which preserves the fibers; this map is $\frac{1}{2}(|x|^2 - |y|^2)$. We can think of this a restriction of the standard T^2 action to the antidiagonal S^1, making xy an S^1-invariant function. The singularity occurs at the unique fixed point of the circle action which is $(0, 0)$ and lives in the $\mu = 0$ level set.

Thus, the base of our fibration has two parameters (r, λ): the radius r and the moment level λ. Spelling it out, the fibration is given by

$$X^0 \to \mathbb{R}^2, (x, y) \mapsto (|xy - \epsilon|, \frac{1}{2}(|x|^2 - |y|^2))$$

However, the singular fiber presents a discontinuity due to the appearance of Maslov 0 disks which splits the moduli space of Lagrangian tori into two chambers, as we will now explain.

Coordinates on the moduli space

Firstly, to specify the coordinates associated to a Lagrangian torus with local system (L, ∇), the precise formulation is that one should choose a reference fiber L_0 and a basis of the homology $H_1(L_0)$, then parallel transport the basis loops to L and measure the symplectic area of the resulting cylinders. The first step is to choose such a basis. We ensure that $H_1(L_0)$ is generated by the circle in the fiber (i.e. the orbit of the S^1 action), which we call γ, as well as the circle coming from the base, which we denote γ'. The coordinates read:

$$(L, \nabla) \mapsto \left(\exp\left(-\int_\Gamma \omega\right)\nabla(\gamma), \exp\left(-\int_{\Gamma'} \omega\right)\nabla(\gamma')\right)$$

To remove dependency on the reference fiber, we can cap off by a disk and consider the weight $z_\beta, \beta \in H_2(X, L)$, instead. We illustrate this in the separate chambers and compute the superpotential:

The case $r < \epsilon$

We fix a reference L_0 and notice that the fibration is trivial in this region. We denote the class of holomorphic disk sections by β whose boundary represents $\partial\beta = -\gamma'$. To remove dependency on the reference fiber L_0, we can cap off with a disk given by filling in the missing fiber:

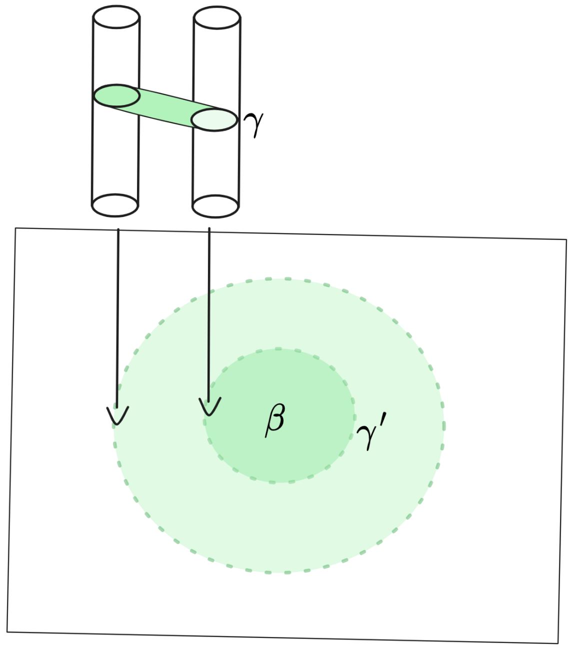


Figure 3: The large circle determines the reference fiber L_0, whereas the small one L. The symplectic area of the cylinder starting from γ is given by the difference of the moment levels. On the other hand, the cylinder corresponding the annulus starting from γ' can be capped off with a disk, whose symplectic area then becomes $\int_\beta \omega$

Capping off amounts to rescaling our coordinates to

$$(w, u) := (T^{-\lambda}\nabla(\gamma), T^{\omega\cdot\beta}\nabla(\gamma'))$$

The coordinate w is associated to the coordinate λ on the base, which should be a globally defined coordinate in both chambers since it comes from a globally defined S^1 action.

We can also think of the disk α representing it as the Lefschetz vanishing thimble on the level 0, starting from a circle on the torus and ending as the singular point. This is in the same homology class as the disk above the singular point in the singular fibre.

However, the disks in class β can bubble off Maslov 0 disks once we increase the radius past the singular fibre and do not give a globally defined coordinate u.

The superpotential in this chamber is just

$$W = u$$

Remark 2.1 (Partial compactifications and superpotential): Note that if we were working with the binodal cubic over the punctured plane, then the superpotential would be empty. However, partially compactifying allows us to use the property that the superpotential is a globally defined, regular function on the mirror, so comparing them on the two sides will tell us how to glue.

For example, another way to show that w is globally defined is by puncturing the plane, but partially compactifying the fibers to be $\mathbb{C}$ instead of $\mathbb{C}^\times$: then, the superpotential in both chambers is just given by w, and since the superpotential is a globally defined regular function on the mirror, we can conclude that w is globally defined. This argument of computing the superpotential of a partial compactifications will be important in the understanding of instanton corrections when we come to the Auroux-Abouzaid-Katzarkov construction.

The case $r > \epsilon$

The tori for large radius can be deformed to the standard product torus, by isotoping them to be centered at the origin without crossing the singular fibre. Hence, they bound two holomorphic disks β_0, β_1, which are still sections over the base, but because they cross the singular fibre, we are presented with the option to cross the x-axis or the y-axis. This follows from a general principle about holomorphic disks bounding standard tori in toric varieties: they are enumerated by the facets of the toric boundary, which in our case is just $\{x = 0\} \cup \{y = 0\}$.

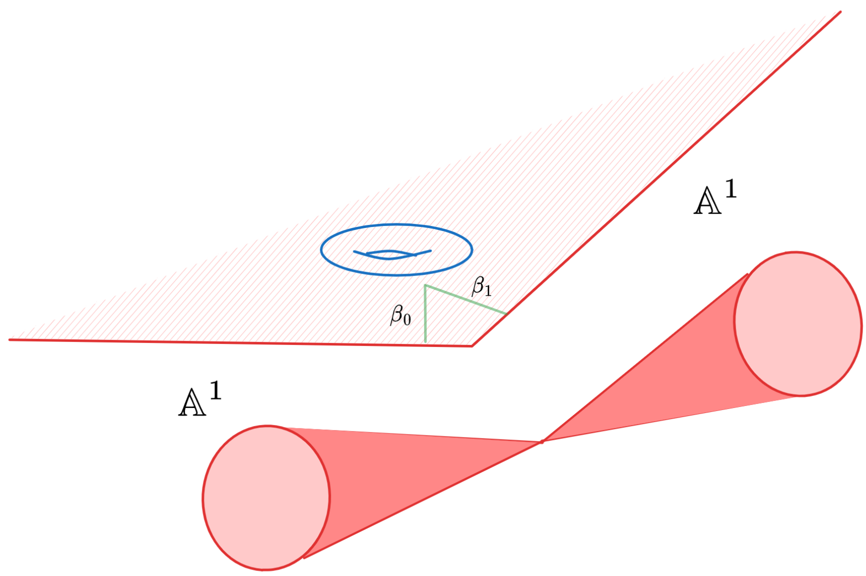


Figure 4: The two disks corresponding to the two facets enumerated by $\{0, 1\}$

To write down a system of coordinates, we identify the torus $L = T_{r,\lambda}$ via its image in the moment polytope of $\mathbb{C}^2$, which is some $(s, \lambda) \in \mathbb{R}^2$. We still call the second coordinate λ, because that is the moment map of the S^1 action, which is given by the moment map of the T^2 action and then projecting to that coordinate. In this case, the symplectic areas of the cylinders with respect to a reference fibers becomes the difference in the coordinates in the moment polytope: if (s_0, λ_0) is the reference, then

$$\int_\Gamma \omega = \lambda_0 - \lambda, \int_{\Gamma'} \omega = s_0 - s$$

This general fact about standard product tori in toric varieties is shown in [ChoOh].

In a similar way as before, we can just rescale to get coordinates

$$(L, \nabla) \mapsto (w, v) := (T^{-\lambda}\nabla(\gamma), T^{-s}\nabla(\gamma'))$$

Importantly, the first coordinate is the same as before! We now need to understand the superpotential and the instanton corrections in order to glue these two chambers.

In general, for each facet $a \in \{0, 1\}$, we have the relation $\partial\beta_a = a\gamma - \gamma'$. In our example, this reduces to

$$\partial\beta_0 = -\gamma', \;\; \partial\beta_1 = \gamma - \gamma'$$

Moreover, the symplectic area of these disks is given by the distance from the point (s, λ) in the moment polytope to the facets, which is $(s, \lambda) \cdot (1, -a) = s - \langle a, \lambda\rangle$. With this in mind, we can express the weights associated to β_0, β_1 using the coordinate system w, v as

$$z_{\beta_0} = \frac{1}{v}, z_{\beta_1} = \frac{w}{v}$$

All in all, the superpotential becomes

$$W = z_{\beta_0} + z_{\beta_1} = \frac{1}{v} + \frac{w}{v}$$

Gluing the chambers

The way to glue the two chambers is to ensure that the superpotential glues to a globally defined function, as in 2.1. In other words, we must glue the chambers (w, u) and (w, v) so that the quantity $W = u = \frac{w+1}{v}$ remains the same. We complete the mirror to obtain

$$(\mathbb{C}^2, D) \leftrightarrow X^\vee = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^\times | uv = w + 1\}, \;\; W = u$$

Remark 2.2 (The binodal cubic is self-mirror): If we were to remove D, then the superpotential disappears on the mirror:

$$\mathbb{C}^2 \setminus D \leftrightarrow \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^\times | uv = w + 1\}$$

But this is the same! Namely, we get all pairs $(u, v) \in \mathbb{C}^2$ such that $w = uv - 1$ is invertible, i.e. the variety $\mathbb{C}^2 \setminus \{uv = 1\}$. In fact, Pascaleff in his thesis [Pasc2014] explores this example and finds a Lagrangian section of the SYZ fibration L such that

$$SH^0 = HW(L, L) = \mathbb{C}[u, v, w][(uv - 1)^{-1}] = \mathrm{Ext}(\mathcal{O}, \mathcal{O})$$

We illustrate wall-crossing with a further example.

2.2 The canonical bundle of $\mathbb{P}^1$

Instead, let's consider the toric variety which is a degeneration of $\mathbb{C}^\times$ that instead of two components has three.

We identify $\mathcal{O}(-2)$ as the GIT quotient $\mathbb{C}^3//\mathbb{C}^\times_{1,1,-2}$ where we remove the locus $x = y = 0$ giving us a projection to $\mathbb{P}^1_{[x:y]}$. The function xyz vanishes to order one on the toric boundary, giving us a fibration $\mathcal{O}(-2) \to \mathbb{C}$ which has a single singular fiber, the toric boundary $\mathbb{A}^1 \cup \mathbb{P}^1 \cup \mathbb{A}^1$, but otherwise is always $\mathbb{C}^\times$. We can construct in the exact same way as before a torus fibration with a single wall, however the gluing will be slightly different. We begin by analyzing the chambers, as before.

Small radius chamber

As before, we have a system of coordinates (w, u) with w the global coordinate and v corresponding to the disk given as a section of the trivial fibration. The superpotential is again

$$W = u.$$

Large radius chamber

In the case $r > 1$ we are again in the standard product torus in a toric variety case, and so the disks are enumerated by the components of the toric boundary. There are three of them and we denote the classes of disks by $\beta_{-1}, \beta_0, \beta_1$ (labelled by the normals to the toric strata).

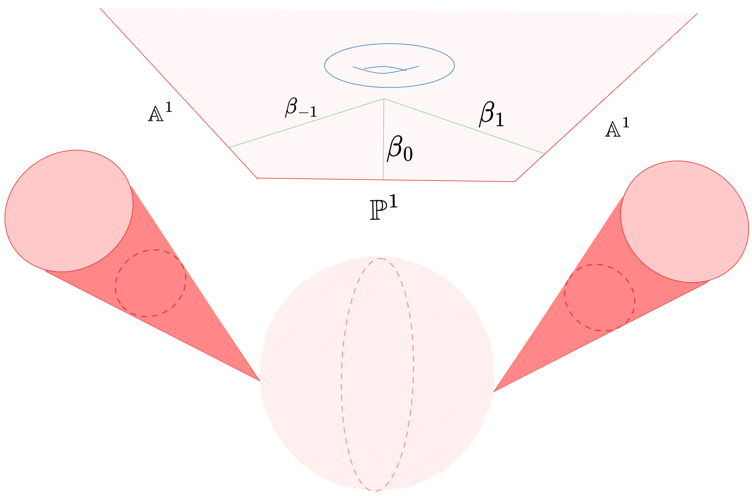


Figure 5: The polytope of $K_{\mathbb{P}^1}$

It is again true that $\partial\beta_a = a\gamma - \gamma'$ and the symplectic areas of the disks β_i are again the distances to the toric boundary in the moment polytope. If we use the coordinate system (w, v) like in the previous example, the weights associated to the three disks are

$$z_{\beta_{-1}} = w^{-1}v^{-1}, z_{\beta_0} = v^{-1}, z_{\beta_1} = wv^{-1}$$

The superpotential becomes

$$W = z_{\beta_{-1}} + z_{\beta_0} + z_{\beta_1} = \frac{w + 1 + w^{-1}}{v}$$

Gluing the two chambers

In order to glue the chambers (w, u) and (w, v) we account for the discrepancy between the superpotentials by setting

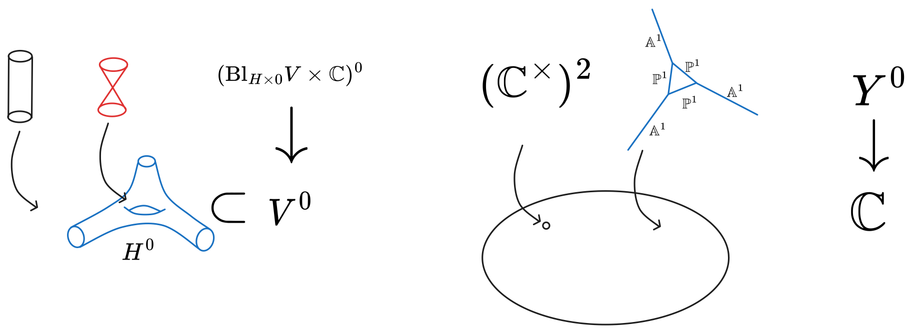
$$W = u = \frac{w + 1 + w^{-1}}{v}$$

The completed mirror should then be the conic fibration

$$K_{\mathbb{P}^1} \leftrightarrow (X^0 = \{(w,u,v) \in \mathbb{C}^\times \times \mathbb{C}^2 | uv = f(w)\}, W = u)$$

where $f(w) = w + 1 + w^{-1}$. This has only two singular fibres at the roots of f. If we were to remove a generic fibre from $K_{\mathbb{P}^1}$, this would remove the superpotential from the mirror X^0. In fact, if we partially compactify X^0 by allowing $w \in \mathbb{C}$, we get the affine quadric $X \simeq T^*S^2$.

Remark 2.3 (Degenerations of $(\mathbb{C}^\times)^n$ and conic fibrations; base-fiber duality): Both of the previous examples, which are toric varieties and thought of as degenerations of $\mathbb{C}^\times$, were found to be mirror to certain conic fibrations. In fact, we will see in Section 3 how to generalize this mirror duality, following [AurAb2015].The idea is that the toric boundary encodes information about the singular locus of the conic fibration, a hypersurface in the base of the mirror. This phenomenon is called *base-fiber duality*, and the picture is as follows:



Base-fiber duality: the thrice-punctured elliptic curve is dual to the toric boundary of $K_{\mathbb{P}^2}$, which appears as the amoeba in the polytope

2.3 The affine quadric

We now do the converse of what we did for the canonical bundle of $\mathbb{P}^1$ and consider an open subset of the blowup at two points of $\mathbb{C}^2$. This will have two wall crossings, and hopefully the mirror polytope should describe the three chambers $\mathbb{C} \times \mathbb{C}^\times$ as half planes which intersect to give the polytope for $K_{\mathbb{P}^1}$ as in Figure 5.

We consider the conic fibration with base $\mathbb{C}^\times$

$$\{(x,y,z) \in \mathbb{C}^2 \times \mathbb{C}^\times | xy = f(z)\}$$

where f is a full quadratic, for example $(z-a)(z-b)$ where $1 < a < b$. The torus fibration is given by considering circles centered at the missing point in $\mathbb{C}^\times$ and lifting them to circles via the S^1 action on the fibers $e^{i\theta} \cdot (x,y,z) = (e^{i\theta}x, e^{-i\theta}y, z)$. We expect to have three chambers, depending on whether $r < a, a < r < b, r > b$, since the singular tori occur at the radii where the conic fibers are singular.

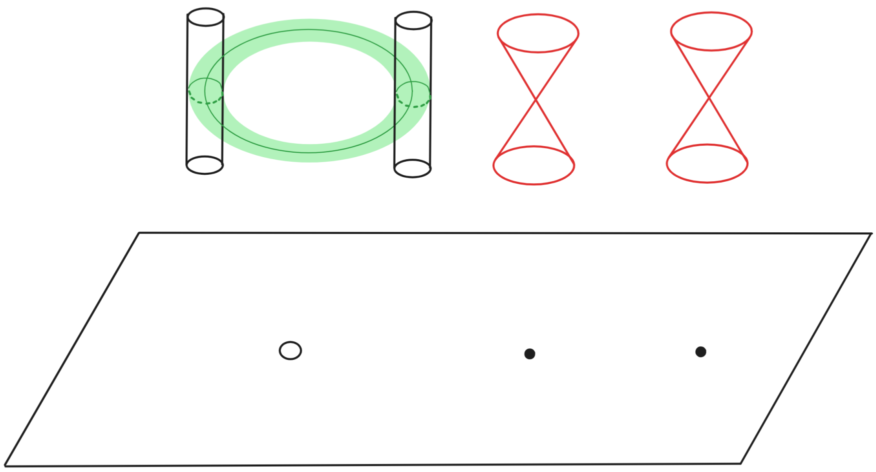


Figure 6: Small radius chamber torus

As before, we have a global coordinate w coming from the S^1 action on the fibers, which corresponds to the moment map $\mu_{S^1} = \frac{1}{2}(|x|^2 - |y|^2)$. The other coordinate is the (exponentiated, complexified) radius $|z|$. Explicitly, we can realize the fibration as

$$X^0 \to \mathbb{R}^2, (x,y,z) \mapsto (\log|z|, \mu(x,y,z))$$

Essentially as before, we have coordinates u_0, u_1, u_2 which correspond to the disk sections once we add in the missing fiber. Under wall crossing, they split off into another such disk, together with a bubbled Maslov 0 disk corresponding to w. The gluing then becomes, as in 1.12

$$u_i = u_{i+1}(1+w), \qquad i = 0,1,2$$

In fact, this is exactly the gluing datum for the toric variety $K_{\mathbb{P}^1}$, except for the fact that we need to make w invertible since it is a globally defined coordinate: thus, the mirror is given by

$$Y^0 = K_{\mathbb{P}^1} \setminus w^{-1}(0)$$

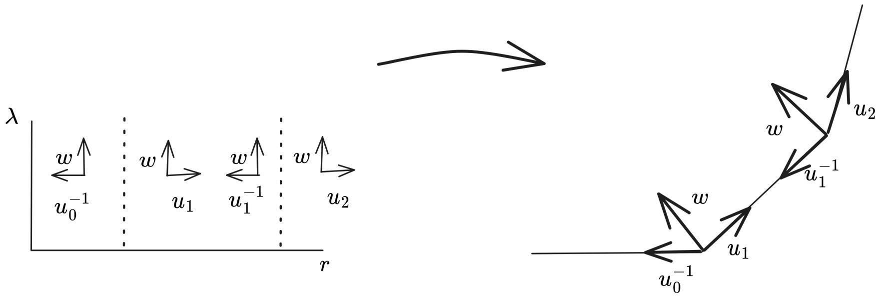


Figure 7: The gluing datum realizes the mirror as a toric variety

Finally, since we are interested in T^*S^2, we need to partially compactify by allowing $z = 0$. This introduces a superpotential on the mirror which enumerates the new disk sections. Thinking of the toric variety as the resolution of the A_1 singularity $st = u^2$, we can think of the superpotential as $W = s$, which is a global function since blowing up the origin does not change the functions on the variety. In GIT coordinates

$$\mathcal{O}(-2) = \mathbb{C}^3_{x,y,z} // \mathbb{C}^\times_{1,1,-2}$$

this is given by the function x^2z (if we can abuse notation and use x, y, z for both spaces). Note that the function xyz vanishing to order 1 on each piece of the toric boundary corresponds to a Maslov 0 disk, whereas the superpotential corresponds to a Maslov 2 disk. This will become important once we need to introduce a grading on the matrix factorizations.

3 The AAK construction

In this chapter, we sketch the general construction of a Lagrangian torus fibration on the conic fibration associated to a hypersurface $H = \mathbb{V}(f)$ in a toric variety V and the recipe for constructing an SYZ mirror, following [AurAb2015].

3.1 From the degeneration to the conic bundle

Let Y be an $n+1$-dimensional toric variety described by a polytope Δ_Y or dually by a fan Σ_Y with primitive generators $(-\alpha_i, 1)$, where α_i are the vertices of Δ_Y.

Proposition 3.1: *There is a Lagrangian torus fibration on $Y^0 = Y \setminus v_0^{-1}(1)$ whose moduli space has two chambers and a single wall. The mirror can be obtained as a conic bundle*

$$X^0 = \{yz = f(x)\}$$

for a suitable f defined using the faces of Δ_Y and a count of curves.

The idea is as follows: we take v_0 to be the function vanishing to order 1 on the toric boundary and remove a generic fiber of v_0 from Y. Each fiber is then an algebraic torus and there is a global T^n action on the space, inherited from the T^{n+1} action as a toric variety. The torus fibration is obtained by taking concentric circles in the base and lifting them via this T^n action. There is a single wall presented once we cross the toric boundary fiber.

Small radius chamber

The coordinates coming from the T^n action $(x_1, \ldots, x_n)$ are global (this can also be shown by using partial compactifications and superpotentials as in 2.1), but there is another one y coming from the circle in the base. It encloses a disk which contributes to the single term in the superpotential

$$W = y$$

Large radius chamber

Now, we can deform to the standard product torus case by moving the circle in the base to be centered at the origin. Again, the coordinates $(x_1, \ldots, x_n)$ are global and there is another coordinate z coming from the disk section over the base. The superpotential can be computed using the product torus disk potential as in [ChoOh], where each facet corresponding to $\alpha \in A$ corresponds to some Maslov 2 disk β_α:

$$W = \sum_{\alpha \in A} z_{\beta_\alpha}$$

However, this is not quite right: one also has to encode stable configurations of disks and spheres. Once this is corrected using a suitable Gromov-Witten invariant κ_α, the result becomes is

$$W = \sum_\alpha (1+\kappa_\alpha) z_{\beta_\alpha}$$

The final step is to express the weights z_{β_α} using the coordinates $(x_1, \ldots, x_n, z)$. In fact, this can be done by noticing that $\partial\beta_\alpha = \sum \alpha_i \gamma_i - \gamma$, where γ_i are the generators for $H_1(T^n)$ in the fibers and γ is the circle in the base. Momentarily setting Novikov parameter to 1, the expression reads

$$W = \sum_\alpha (1+\kappa_\alpha) \frac{x_1^{\alpha_1} \ldots x_n^{\alpha_n}}{z}$$

Gluing the two chambers

The result of identifying the two chambers by ensuring that the superpotentials glue to a globally defined function necessitates

$$y = \sum_\alpha (1+\kappa_\alpha) \frac{\mathbf{x}^\alpha}{z}$$

Hence, we can express the mirror (after completing) as

$$X^0 = \{(\mathbf{x}, y, z) \in (\mathbb{C}^\times)^n \times \mathbb{C}^2 | yz = f(\mathbf{x})\}$$

where f is the polynomial defined using the GW invariants above.

Remark 3.2 (Novikov parameter): We have been a bit sloppy when it comes to the Novikov parameter here: formally, one should get a family of mirrors parametrized via a Novikov parameter T, or equivalently a variety over the Novikov field $\mathbb{K}$ defined by

$$\mathbb{X}^0 := \{yz = \tilde{f}(\mathbf{x}), \tilde{f}(\mathbf{x}) = \sum_\alpha (1+\kappa_\alpha) T^{\rho(\alpha)} \mathbf{x}^\alpha\}$$

3.2 From the blowup to the degeneration

In section 1, we explained how to construct mirrors to Fano toric varieties: this is well-understood since the work of [ChoOh]. However, in the previous section, we sketched how certain toric degenerations of algebraic tori are mirror to conic fibrations. The importance of these conic fibrations is because they encode information about the hypersurfaces $H = \mathbf{V}(f)$.

While finding a Lagrangian torus fibration for a toric variety is easy, by using the moment map, the same does not hold for hypersurfaces. In order to deal with this, one passes to a bigger space, namely the deformation to the normal cone $\mathrm{Bl}_{H\times 0} V \times \mathbb{C}$. An open subset of this is precisely the conic fibration X^0 we saw as a mirror in the previous section.

In fact, the Fukaya category of this blowup relative to the projection to $\mathbb{C}$ will be the same as that of H, whereas a theorem by Bondal and Orlov assures us that the derived category of the blowup of $H \times 0 \subset V \times \mathbb{C}$ admits a semiorthogonal decomposition in terms of those of H and $V \times \mathbb{C}$. So this will give us interesting information in terms of homological mirror symmetry as well.

In the following section, we will describe a Lagrangian torus fibration on this blowup construction, and then sketch how to construct its SYZ mirror. In short, the argument goes as follows:

Observation : *The essential idea boils down to using an S^1 action on the fibers of the projection to V, and then lifting the standard torus fibration on V to X by spinning via this action. This needs to be modified slightly by carefully choosing a Kahler form and identifying the reduced spaces with V in an equivariant way. Once this is done, one sees that most Lagrangian tori project to standard product tori in V and can be understood by reduction to the toric case. The problematic ones are those whose projection meets the hypersurface H. This can be visualized in the conic fibration, where the fibers are singular only over H, and hence the tori which bound holomorphic disks are the singular ones where circles get squished to points. This describes a wall-chamber decomposition, where the walls are given by the tropicalization of the hypersurface H. The gluing datum then is calculated by carefully choosing partial compactifications and computing superpotentials, as in 2.1. The resulting SYZ mirror has the gluing data precisely of the toric variety associated to the Newton polytope of f.*

The blowup construction

The setup is the following: we equip V with a toric Kahler form ω_V. The toric action of the torus has moment map $\mu_V : V \to \mathbb{R}^n$. The image is the Delzant polytope Δ_V which is dual to the fan Σ_V defining V. The logarithm map

$$\mathrm{Log}_\tau : x \mapsto \frac{1}{|\log \tau|}(\log|x_1|, \ldots, \log|x_n|)$$

is related to the moment map by a diffeomorphism g_τ.

The hypersurface H is given by the zero locus of a section of a line bundle $\mathcal{L}$. Hence, by adjunction, the normal bundle of $H \times 0 \subset V \times \mathbb{C}$ is $\mathcal{L} \oplus \mathcal{O}$. Let us write $\mathbf{x}$ for elements of V and y for elements of the $\mathbb{C}$ factor. The blowup X is a hypersurface in the projective bundle

$$X = \{\mathbf{x}, y, [u : v] \in \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \mid f(\mathbf{x})v = yu\}$$

We can lift the circle action on $V \times \mathbb{C}$ which rotates the $\mathbb{C}$ factor lifts to X by rotating both y and v by $e^{i\theta}$. This has two types of fixed points:

- The set where $y = v = 0$ which is the proper transform of $V \times 0$, i.e. $\tilde{V} = \overline{p^{-1}(V \times \mathbb{C} \setminus H \times 0)}$
- The other set is a section $\tilde{H}$ in the projectivized normal bundle on $H \times 0$ given by projectivizing the zero section plus the trivial section: in each fiber, it looks like $[0 : 1]$.

To do SYZ, we need a holomorphic $n+1$-form on an open Calabi-Yau piece $X^0 \subset X$. To do that, we take the holomorphic form on the open stratum $V^0 \times \mathbb{C}^\times$ which is

$$i^{n+1} \prod d \log x_j \wedge d \log y$$

This has poles along $V \times 0$ and $D_V \times \mathbb{C}$, where D_V is the toric boundary. Pulling this back to X, the poles occur along $\tilde{V}$, the proper transform of V, as well as the preimage of the toric boundary of V, i.e. $p^{-1}(D_V \times \mathbb{C})$. When we remove this divisor $D = \tilde{V} \cup p^{-1}(D_V \times \mathbb{C})$, we see that

$$X^0 := X \setminus D = \{\mathbf{x}, y, z \in \mathcal{O} \oplus \mathcal{L} | f(\mathbf{x}) = yz\}$$

is a conic bundle over the stratum $V^0 = (\mathbb{C}^\times)^n$.

The Kahler form

To equip this with a Kahler form, take a Hermitian metric on $\mathcal{L}$ and get a Kahler form on $\mathbb{P}(\mathcal{L} \oplus \mathcal{O})$ living in $c_1(T)$ where T is the tautological bundle. This should give $\frac{i}{2\pi}\partial\overline{\partial}\log(|u|^2 + |v|^2) = \frac{i}{2\pi}\partial\overline{\partial}\log(|f(\mathbf{x})|^2 + |y|^2)$, since $f(x)v = yu$. The idea is that on X, we interpolate between $p^*\omega_{V\times\mathbb{C}}$ outside of the exceptional divisor, where p is a biholomorphism and

$$p^*\omega_{V\times\mathbb{C}} + \frac{i\epsilon}{2\pi}\partial\overline{\partial}\log(|f(x)|^2 + |y|^2)$$

over the exceptional divisor, so as to ensure the symplectic area of the $\mathbb{P}^1$ fibers are small, equal to ϵ.

To do this, we use an S^1-invariant cutoff function $\chi(x, y)$, supported in a tubular neighbourhood of $H \times 0$ and insert it inside:

$$\omega_\epsilon := p^*\omega_{V\times\mathbb{C}} + \frac{i\epsilon}{2\pi}\partial\overline{\partial}\chi\log(|f(x)|^2 + |y|^2)$$

The reason is that the blowup takes a symplectic tubular neighbourhood of $H \times 0$ and collapses the sphere bundle of the normal bundle to $\mathbb{CP}^1$ via the Hopf map. This makes the projection $p : X \to V \times \mathbb{C}$ a symplectomorphism outside a neighbourhood of the exceptional divisor E.

The Lagrangian torus fibration

We want to construct an S^1-invariant Lagrangian torus fibration. Hence, the torus fibers should be contained in the level sets of the moment map.

The moment map for the circle action is essentially area:

$$\mu(x, y) = \int_{\{x\}\times D(|y|)} \omega_\epsilon$$

where we integrate over a disk bounding an orbit $(x, e^{i\theta}y)$. Applying Stokes' theorem via the explicit formula, we see that

$$\mu(x, y) = \begin{cases} \pi|y|^2 + \epsilon\frac{|y|^2}{|f(x)|^2+|y|^2} \text{ near E} \\ \pi|y|^2 \text{ away from E} \end{cases}$$

The critical points of μ are the fixed points of the circle action, which occur on $\tilde{V} = \{y = v = 0\} = \mu^{-1}(0)$ and also on $\tilde{H} \subset \mu^{-1}(\epsilon)$ (this is the symplectic area of the $\mathbb{P}^1$ fibers). Hence, all the other level sets are smooth.

We have that for any $\lambda > 0$, the level set $\mu^{-1}(\lambda)$ intersects $p^{-1}(\{x\} \times \mathbb{C})$ exactly along an S^1 orbit , since μ is an increasing function of the modulus $|y|$. So we can identify

$$X_{red,\lambda} \simeq V$$

via the projection to the first factor. Equivalently, the idea is that $x \in V$ maps to the unique S^1 orbit in $p^{-1}(\{x\} \times \mathbb{C})$ of height λ.

Far away from $\lambda = \epsilon$, the isomorphism respects the Kahler forms: $\omega_V = \omega_{red,\lambda}$. As we get closer to ϵ, the reduced form differs, but is cohomologous to the standard one. For $\lambda < \epsilon, [\omega_{red,\lambda}] = [\omega_V] - \max(0, \epsilon - \lambda)[H]$. The problem is that this is not T^n invariant, and a certain modification needs to be made before we pass to the standard torus fibration on V.

Auroux-Abouzaid-Katzarkov [AurAb2015] prove that there is a family of diffeomorphisms $\phi_\lambda : X_{red,\lambda} \simeq V$ intertwining $\omega_{red,\lambda}$ and the toric Kahler form on V. These can then be used to form a Lagrangian torus fibration

$$\pi_\lambda : X^0_{red,\lambda} \xrightarrow{\phi_\lambda} V^0 \xrightarrow{\mathrm{Log}_\tau} \mathbb{R}^n$$

From the reduced spaces to the whole

We can obtain a global Lagrangian torus fibration by reducing the circles given by the S^1-action in the fibers and using the standard torus fibration for V. The map is defined as

$$X^0 = \bigcup_\lambda \mu^{-1}(\lambda) \to \bigcup_\lambda X^0_{red,\lambda} \xrightarrow{\pi_\lambda} \bigcup_\lambda \mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}_+ = B$$

which sends $x \mapsto (\pi_\lambda(\overline{x}), \lambda)$. This is singular along $\lambda = \epsilon$. We will obtain affine charts outside a set of walls, which are codimension 1 cuts.

The fibers are smooth Lagrangian tori, unless $\lambda = \epsilon$, which is the level where the singular points in the conics appear. Since $\lambda > 0$ the fixed points occur only along the section $\tilde{H}$. Under the logarithm map, this corresponds to $\Pi' \times \{\epsilon\} = B_{sing}$, the set of singular fibers.

Constructing the SYZ mirror

The main idea now is that the only Lagrangian tori in the SYZ fibration bounding holomorphic disks are those that intersect $p^{-1}(H \times \mathbb{C})$ under the projection $p : X^0 \to V^0 \times C$. These are the singular tori where one of the S^1 orbits is a fixed point, and the disks they bound has Maslov index 0 whose boundary represents the class of the S^1-orbit. One can project down to V and use the maximum principle to ensure that the disks must live inside fibers. Hence if the fiber is a smooth conic $\mathbb{C}^\times$, the disk must be constant, but if it is a singular conic, there are in fact holomorphic disks.

These singular tori present walls in the moduli space and thus it has as many chambers as the amount of connected components of $\mathbb{R}^n \setminus \mathrm{Log}(U_H)$, the complement of the amoeba of the hypersurface over which the obstructed tori live.

We combine all of this into a proposition:

Proposition 3.3 (Chambers of the Lagrangian torus fibration): *The SYZ fibration for X^0 has chambers U_α enumerated by $\alpha \in A$, the connected components of the complement of the amoeba of H. This is the same as the number of facets of the Newton polytope of the defining function of H. The unobstructed tori in each of these chambers (i.e. those which do not meet $p^{-1}(H \times \mathbb{C})$) project to standard product tori in $V \times \mathbb{C}$.*

The rest of the job reduces to understanding the gluing data for these chambers, which can be shown to be exactly the gluing data for the toric variety defined by the Newton polytope of H.

There is a set of coordinates on $U_\alpha^\vee$ given by

$$(v_{\alpha,1}, \dots, v_{\alpha,n}, w_{\alpha,0})$$

where w_0 is the global coordinate coming from the global S^1 action (we will justify this in a moment). The trick to understand the gluings comes from using the fact that the superpotential is a regular function on the mirror, so we could partially compactify and compare, and also use 1.12. This will also help in understanding the superpotential for the mirror to X.

The proof proceeds in the following steps:

- First, one could partially compactify X^0 to $X^0_+ = p^{-1}(V^0 \times \mathbb{C})$. This partial compactification completes the conic fibration to have generic fibers $\mathbb{C}$ instead of $\mathbb{C}^\times$, which is where the holomorphic disks come in. The superpotential becomes, in any chamber whatsoever,

  $$W_{X^0_+} = w_0$$

  This shows that the function w_0 is globally defined in all chambers.

- To understand the other coordinates v_i, one could also partially compactify X^0 to X_σ by enlarging V^0 by adding in a piece D_σ of the toric boundary corresponding to some $\sigma \in \mathbb{Z}^n$ a primitive generator in the fan Σ_V such that the facet is given by the equation $\langle \sigma, u \rangle + \varpi = 0$. If we denote by $\mathbf{v}_\alpha^\sigma := v_1^{\sigma_1} \dots v_n^{\sigma_n}$, then the superpotential for $(L, \nabla) \in U_\alpha^\vee$ becomes

  $$W_{X_\sigma} = (1 + T^{-\epsilon} w_0)^k T^{\varpi} \mathbf{v}_\alpha^\sigma \tag{3.1}$$

  signifying the bubbling of exceptional disks of area ϵ. The exponent k is an intersection number and is equal to $\langle \alpha - \alpha_{min}, \sigma \rangle$.

  Essentially, this is done by reduction to the case $V^0 \cup D^0_\sigma = \mathbb{C} \times (\mathbb{C}^\times)^{n-1}, \sigma = (1, 0, \dots, 0)$. The composition of the holomorphic disk u with the projection to $\mathbb{C} \times (\mathbb{C}^\times)^{n-1} \times \mathbb{C}$ must then look like $z \mapsto (r_1 z, x_2, \dots, x_n, r_0 \gamma(z))$ by using the maximum principle for all the $\mathbb{C}^\times$ factors and the fact that the disk has Maslov index 2, i.e. it intersects the divisor once and so the first projection is a biholomorphism. There are constraints on γ: it can only vanish when the rest $(r_1 z, x_2, \ldots, x_n) \in H$ and moreover the intersection number between the disc and H is $k = \langle \alpha - \alpha_{min}, \sigma \rangle$. Supposing that the intersection set is transverse and consists of points $\{a_1, \ldots, a_k\}$ then for any subset $I \subset \{1, 2, \ldots, k\}$ we have a Blaschke product

$$\gamma_I := \prod \frac{z - a_i}{1 - \overline{a_i} z}$$

Then $\gamma_I^{-1} \gamma$ must be a constant map, as it has no zeros and maps the unit circle to itself. So we conclude that there are 2^k disks contributing to the superpotential. The weights of these disks β_I in the coordinates for $U_\alpha^\vee$ can be shown to be given by

$$z_{\beta_I} = (T^{-\epsilon} w_0)^{|I|} T^{\omega} v_{\alpha,1}$$

Summing them up, we get the binomial expansion for 3.1

- This same result can be generalized to arbitrary monomials $\mathbf{v}_\alpha^\sigma$, where $\sigma \in \mathbb{Z}^n$ is any primitive vector.

- Finally, since the superpotential must glue to a regular function on the whole of the mirror, this then implies that the correct gluing across a wall between U_α and U_β is given by

$$\mathbf{v}_\alpha^\sigma = (1 + T^{-\epsilon} w_0)^{\langle \alpha - \beta, \sigma \rangle} \mathbf{v}_\beta^\sigma$$

We summarize all of this into a theorem, which is too big to fit into the rest of this page:

Theorem 3.4 (Auroux-Abouzaid-Katzarkov): *The conic bundle X^0 admits an SYZ fibration with chambers enumerated by the connected components of the complement of the amoeba of the hypersurface H. The instanton corrections give a set of gluing data that precisely coincides with the charts for the toric variety Y defined by the Newton polytope*

$$\Delta_Y = \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R} | \eta \geq \mathrm{trop}(f)(\xi)\}$$

which is a degeneration under w_0 with generic fiber an algebraic torus and central fiber given by the boundary of the polytope, which can be identified with the tropicalization of H. Hence, the completed SYZ mirror of X^0 is the subset of Y where the global function w_0 is nonvanishing:

$$X^0 \leftrightarrow Y^0 := Y \setminus w_0^{-1}(0)$$

Once we compactify to X by adding in all the toric divisors in the base (corresponding to the terms W_{X_σ}) together with compactifying the fibers (corresponding to $W_{X_+^0}$), the mirror becomes eqipped with a superpotential:

$$X \leftrightarrow (Y^0, W_0 = W_{X_+^0} + \sum W_{X_\sigma} = w_0 + \sum (1 + T^{-\epsilon} w_0)^{\langle \alpha - \alpha_i, \sigma \rangle} T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i})$$

Finally, in the case that V is affine, one also obtains an SYZ mirror to the hypersurface H by considering a slightly different superpotential:

$$H \leftrightarrow (Y, W_0^H = -v_0 + \sum W_{X_\sigma}$$

where $w_0 = -T^\epsilon + T^\epsilon v_0$).

The last part of the theorem requires further justification, in particular the fact that the Fukaya category of H faithfully embeds into the Fukaya category of X relative to W. For more details on this, we refer to Section 7 of [AurAb2015] and the sequel paper [AurAb2023].

We invoke the base-fiber duality picture again, so as to visialize the theorem:

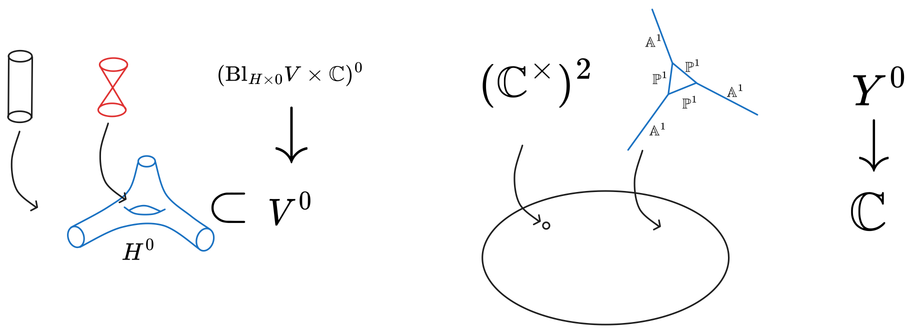


Figure 8: The mirror to the punctured elliptic curve is the canonical bundle of $\mathbb{P}^2$ with a suitable superpotential

3.3 Example: the canonical bundle of projective space

We aim to generalize the situation of $K_{\mathbb{P}^1}$ to all dimensions, by using the AAK recipe. The idea is that the polytope for $K_{\mathbb{P}^n}$ has $n+2$ faces, so we seek a degree $n+1$ polynomial with $n+2$ terms.

Consider the family of hypersurfaces defined by

$$f = t(x_1 + \dots + x_n + \frac{1}{x_1 \dots x_n}) + 1$$

The polytope Δ_Y defined by the tropicalization $\phi(\xi) = \max\{\xi_1 - 1, \xi_2 - 1, \dots, \xi_n - 1, -\xi_1 - \dots - \xi_n - 1, 0\}$ can also be defined by its fan $\Sigma_Y \subset \mathbb{R}^{n+1}$ which is spanned by the vectors $(-\alpha, 1)$ which in our case are

$$\{v_0, \dots, v_{n+1}\} = \{(-1,0,\dots,0,1), (0,-1,\dots,0,1), \dots, (0,\dots,-1,1), (-1,-1,\dots,-1,1), (0,\dots,0,1)\}$$

We can realize the toric variety as a quotient of $\mathbb{C}^{n+2}$ in two steps. Firstly, we throw away the locus corresponding to subsets of $\{v_0, \dots, v_{n+2}\}$ which do not generate a cone: this is precisely $z_0 = \dots = z_{n+1} = 0$.

Then, we find the subgroup given as a kernel of the matrix

$$\begin{pmatrix} -1 & 0 & \dots & 0 & 1 & 0 \\ 0 & -1 & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix}$$

We see that it is precisely the subgroup $\langle (1,1,\dots,1,-n-1) \rangle \subset \mathbb{Z}^{n+2}$ which generates the one-parameter subgroup $\mathbb{C}^{\times}_{1,1,\dots,1,-n-1}$. We recover the variety

$$\frac{\mathbb{C}^{n+2} \setminus \{z_0 = \dots = z_{n+1} = 0\}}{\mathbb{C}^{\times}_{1,1,\dots,1,-n-1}} \simeq K_{\mathbb{P}^n}$$

The recipe that AAK provide tells us that

Proposition 3.5:

- *The open Calabi-Yau manifold Y^0 obtained from removing a generic $(\mathbb{C}^{\times})^n$ fiber from $K_{\mathbb{P}^n}$ is SYZ mirror to the open conic bundle $X^0 = \{(\mathbf{x}, y, z) \in (\mathbb{C}^{\times})^n \times \mathbb{C}^2 | yz = f(\mathbf{x})\}$.*

- *$K_{\mathbb{P}^n}$ is SYZ mirror to the Landau-Ginzburg model (X^0, y).*

- *The Landau-Ginzburg model (Y^0, W_0) is SYZ mirror to X*

- *The Landau-Ginzburg model $(K_{\mathbb{P}^n}, W_0^H = -z_1 \dots z_{n+2})$ is SYZ mirror to the hypersurface $H = \mathbb{V}(f) \subset (\mathbb{C}^{\times})^n$.*

4 An exercise in homological mirror symmetry

In this section, we review the oft-stated mirror relation between T^*S^2 and $K_{\mathbb{P}^1}$. This naive statement is wrong: it has to be modified by passing to open subsets of both. Once we compactify one, a superpotential is added to the other. We will review both options, ideally producing equivalences

$$\mathcal{W}(T^*S^2) \simeq \mathrm{MF}(K_{\mathbb{P}^1}, x^2 z)$$
$$FS((T^*S^2)^0, y) \simeq D^b(K_{\mathbb{P}^1})$$

4.1 The open case

We begin with the open subset X^0 from the blowup construction, which is the conic fibration

$$X^0 := \{(x,y,z) \in \mathbb{C}^\times \times \mathbb{C}^2 | yz = f(x)\}$$

The function f is chosen to have two roots and we get an open subset of the affine quadric, mirror to $K_{\mathbb{P}^1} \setminus \mathbb{C}^\times$, as we saw in 2.3.

SYZ transforms

We begin by describing the SYZ transforms of the zero section and the cotangent fiber.

The zero section Firstly, we identify $T^*S^2 \simeq Q = \{z_1^2 + z_2^2 + z_3^2 = 1\}$:

$$(x, v | \langle x, v \rangle = 0) \in T^*S^2 \mapsto x \cosh(|v|) + i \frac{\sinh(|v|)}{|v|} v \in Q$$

The inverse map is

$$z \in Q \mapsto x = \mathfrak{Re}(x), y = \mathfrak{Im}(z) \mapsto (|x|^{-1} x, |x| y) \in T^*S^2$$

We see that the zero section is identified with the real locus $y = 0$ i.e. $z_1, z_2, z_3 \in \mathbb{R}$. Under the projection to z_3 this fibers over the path connecting the singular values, and we see S^2 as the blue matching sphere:

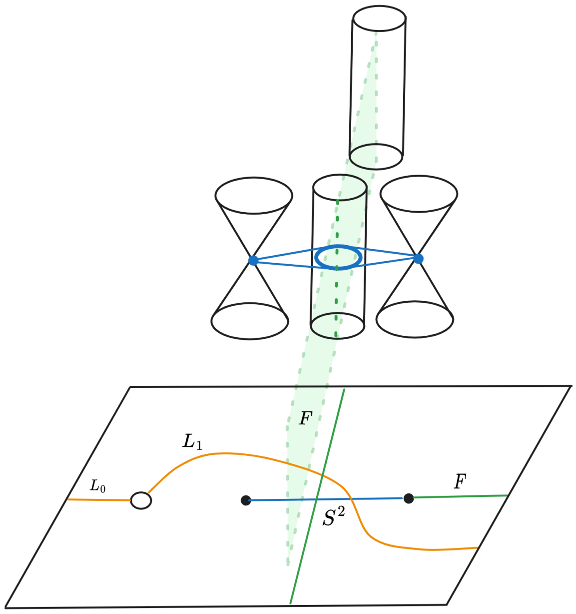


Figure 9: Zero section is the matching sphere in blue. Cotangent fibre over point in equatorial circle is the plane F. The cotangent fiber at infinity is F'.

To understand the support of its mirror, we need to think about the Lagrangian tori it meets. They are precisely those with $r \in (a,b)$ and $\lambda = 0$, which describes exactly $\mathbb{P}^1$ on the mirror. This is because more generally, the tori of level 0 describe the toric boundary, glued in the way that the three chambers are glued (given as the half-planes in the toric picture).

The tori intersect cleanly with the zero section along an S^1 and we calculate the Floer cohomology, using the Morse-Bott, or Pozniak, spectral sequence:

$$HF^\bullet(S^2, T_{r,0}) \simeq H^\bullet(S^1) = \mathbb{C}[0] \oplus \mathbb{C}[1]$$

Correspondingly, the most natural mirror $\iota_*\mathcal{O}_{\mathbb{P}^1}$ has the following:

$$\mathrm{Ext}^\bullet(\iota_*\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_p) = \begin{cases} \mathbb{C}[0] \oplus \mathbb{C}[1], & p \in \mathbb{P}^1 \\ \\ 0 \quad \text{otherwise} \end{cases}$$

This computation can be done using a Koszul resolution: we think of $\mathbb{P}^1 = \mathbb{V}(z)$ in the GIT quotient $\mathbb{C}^3 // \mathbb{C}^*_{1,1,-2}$ and hence get

$$0 \to \mathcal{O}(-2) \xrightarrow{z} \mathcal{O} \to \iota_*\mathcal{O}_{\mathbb{P}^1} \to 0$$

When we hom into $\mathcal{O}_p$, we get the complex

$$\mathbb{C} \xrightarrow{z(p)} \mathbb{C}$$

whose cohomology is exactly the one prescribed above. This is consistent with Theorem 4.1 in [Chan13], as the path between the singular values has winding number 0.

The cotangent fibre Now, we try to understand the cotangent fibres. For example, the one over the point $(0,0,1) \in S^2$. The points in this cotangent fiber F' correspond to

$$((0,0,1) \in S^2, v = (v_1, v_2, 0)) \in T^*S^2 \mapsto x\cosh(|v|) + i\frac{\sinh(|v|)}{|v|}v \in \mathbb{C}^3 \mapsto \cosh(|v|) \in \mathbb{C}$$

after projecting to the third coordinate. So this lies above the real line.

We could also consider the fiber over a point with last coordinate 0, which would live over the imaginary line. These we visualize as the fiber F.

We see that the support of the mirror of F' can only lie in the third chamber, as it only intersects the tori which encircle both singular fibres. We claim that the mirror is in fact $\mathcal{O}_{\mathbb{A}^1}$ where $\mathbb{A}^1 = \mathbb{V}(x)$. Ideally, when we compute $HF(F,T)$ for a torus with big radius, these have two intersection points. If the level of T is nonzero, hopefully the differential becomes a unit, because the symplectic areas of the two contributing disks are different. So the hypersurface is just the $\mathbb{A}^1$ which is the fiber at infinity in $\mathcal{O}(-2)$, and can be thought of as $\mathbb{V}(x)$.

The Lagrangian sections There are also two other noncompact Lagrangians, arising as sections of the SYZ fibration and whose mirrors are line bundles on the total space. We take the 'strongly' admissible paths γ_0, γ_1 (the orange ones in the picture) defining Lagrangian sections L_0, L_1 which are mirror to $\mathcal{O}, \mathcal{O}(1)$ (Theorem 1.1. in [ChanUeda13]). In fact, these line bundles generate $D^b(K_{\mathbb{P}^1})$, which we will explain later.

We can apply surgery to these two Lagrangians, the result being the cotangent fiber. In categorical terms, this is a cone:

$$F = \mathrm{cone}(L_0 \to L_1).$$

On the mirror, we expect

$$F^\vee = \mathrm{cone}(\mathcal{O} \to \mathcal{O}(1)).$$

4.2 Mirror symmetry for T^*S^2

Now, we compactify the A-model to the affine quadric and introduce a superpotential on the B-model. In terms of the coordinates as the resolution of the A_1 singularity $st = u^2$, it is expressible as $W = s$ and equivalently in x, y, z coordinates $s = x^2 z, t = y^2 z, u = xyz$. So $W = x^2 z$. By theorems of Abouzaid [Ab11] and Viterbo [Vit] and by using a James splitting for the loop space of S^2, one has

$$\mathcal{W}(T^*S^2) \text{ is generated by a cotangent fiber } F, \text{ and } HW(F,F) \simeq SH^\bullet(T^*S^2) \simeq H_{-\bullet}(\mathcal{L}S^2) \simeq \mathbb{C}[t]$$

where $|t| = -1$. So it remains to show that the mirror to the cotangent fiber, which is the matrix factorization corresponding to $\mathcal{O}_{\mathbb{V}(x)}$, generates. We moreover calculate its endomorphism algebra in order to verify HMS.

Computation of matrix factorizations If we want to argue directly in the matrix factorization category, we need to introduce an R-charge, which is a way to introduce a grading. In particular, it is given by the action $\mathbb{C}^{\times}_{2,0,-2}$ instead of $\mathbb{C}^{\times}_{1,1,-2}$ and it makes the superpotential have weight 2. The reason we take this R-charge is because xyz should have weight 0 as it corresponds to a Maslov 0 disk, whereas x^2z should have weight 2 as it corresponds to a Maslov 2 disk. We are presented with any option of the form $(n, n-2, 2-2n)$ but this is defined modulo the $(1,1,-2)$ action, so we pick $(2,0,-2)$.

A character of this action produces a line bundle, tensoring with which gives the shift functor for the graded matrix factorizations (see [Segal11]). Notice that the critical locus of $W = x^2z$ is inside the affine subset $\mathbb{A}^2_{x,z}$ so there are no higher derived Exts. The matrix factorization corresponding to $\mathcal{O}_{x=0}$ is the one given (via ideal sheaf sequence) by

$$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1)[-1]$$

which can be thought of naively as the matrix factorization $x.xz = x^2z$.

Proposition 4.1 (Matrix factorizations): *The matrix factorizations category* $\mathrm{MF}(Y^0, x^2z)$ *is generated by a single object whose endomorphism algebra is* $\mathbb{C}[t]$ *with* $|t| = -1$

Proof. For generation, we use Knorrer periodicity, as in for example [Ship2012]: the critical locus of the superpotential is supported inside an affine plane and we have

$$\mathrm{MF}(\mathbb{A}^2_{x,z}, x^2z) \simeq D^b(\mathbb{C}[x]/x^2)$$

The zero section corresponds to $\mathbb{V}(z)$ which in turn gives the matrix factorization $x^2.z$ and this corresponds to $\mathbb{C}[x]/x^2$. Similarly, the cotangent fibre corresponds to $\mathbb{V}(x)$ which gives the matrix factorization $x.xz$ and corresponds under Knorrer periodicity to $M = \mathbb{C}[x]/x$, which has the infinite resolution

$$\cdots \to R \xrightarrow{x} R \xrightarrow{x} \mathbb{C}[x]/x \to 0$$

where $R = \mathbb{C}[x]/x^2$. We see that $R = \mathrm{cone}(M \to M[-1])$. By thinking about Jordan block decompositions of vector spaces with square-zero action of x, we see that M generates.

It remains to show the endomorphism algebra is the one above. We do this in the odd and even cases separately.

Odd case We study $\mathrm{Ext}^i(\mathcal{E},\mathcal{E})$ by examining the morphisms $\mathrm{Hom}(\mathcal{E},\mathcal{E}[i])$ which form the closed maps in the hom-chain complex $\overline{Hom}(\mathcal{E},\mathcal{E}[i])$ and mod out by chain homotopies, which are the exact maps. The resulting cohomology is exactly $\mathrm{Ext}^i(\mathcal{E},\mathcal{E})$. For the odd case, we look at the diagram:

$$
\begin{array}{ccccccc}
\mathcal{O}(-1)[-1] & \xrightarrow{\;x\;} & \mathcal{O}[1] & \xrightarrow{\;xz\;} & \mathcal{O}(-1)[1] & \qquad & \mathcal{E} \\
f\downarrow & \overset{h}{\swarrow} & g\downarrow & \overset{h'}{\swarrow} & \downarrow f & & \\
\mathcal{O}[2k-1] & \xrightarrow[xz]{} & \mathcal{O}(-1)[2k-1] & \xrightarrow[x]{} & \mathcal{O}[2k+1] & \qquad & \mathcal{E}[2k-1]
\end{array}
$$

The monomial basis for the f's is enumerated by $x^\alpha y^{\alpha-2k+1} z^{\alpha-k}$ whereas the one for g is $x^\alpha y^{\alpha-2k+1} z^{\alpha-k+1}$. The chain homotopies have a monomial basis $x^{\alpha'} y^{\alpha'-2k+2} z^{\alpha'-k+1}$. When $k < 0$ there are no constraints on α and we immediately see that

$$
\mathrm{Ext}^k(\mathcal{E},\mathcal{E}) = \frac{\langle x^\alpha y^{\alpha-2k+1} z^{\alpha-k+1} \rangle_{\alpha \geq 0}}{xz \langle x^{\alpha'} y^{\alpha'-2k+2} z^{\alpha'-k+1} \rangle_{\alpha' \geq 0}} = \mathbb{C} \cdot (f = y^{-2k+1} z^{-k}, g = y^{-2k+1} z^{-k+1})
$$

The generator corresponds to $\alpha = 0$. Moreover, we see that the element $t = (y, yz)$ has $t^{1-2k} = (y^{-2k+1} z^{-k}, y^{-2k+1} z^{-k+1})$ and is in degree -1.

However, when $k > 0$, we must constrain $\alpha \geq 2k-1, \alpha' \geq 2k-2$. When we do this, the generator corresponding to $\alpha = 0$ no longer survives and there are as many closed maps as there are exact ones, and hence

$$
\mathrm{Ext}^{2k-1}(\mathcal{E},\mathcal{E}) = 0, k > 0
$$

Even case We do the same for the even case, where the diagram is shifted.

$$
\begin{array}{ccccccc}
\mathcal{O}(-1)[-1] & \xrightarrow{\;x\;} & \mathcal{O}[1] & \xrightarrow{\;xz\;} & \mathcal{O}(-1)[1] & \qquad & \mathcal{E} \\
f\downarrow & \overset{h}{\swarrow} & f\downarrow & \overset{h'}{\swarrow} & \downarrow f & & \\
\mathcal{O}(-1)[2k-1] & \xrightarrow[x]{} & \mathcal{O}[2k+1] & \xrightarrow[xz]{} & \mathcal{O}(-1)[2k+1] & \qquad & \mathcal{E}[2k]
\end{array}
$$

The f's have a basis $x^\alpha y^{\alpha-2k} z^{\alpha-k}$ whereas the h's have a basis of monomials $x^{\alpha'} y^{\alpha'-2k+1} z^{\alpha'-k+1}$.

When $k < 0$, we have no constraint on α and we see that we can write the basis for closed f as $(xyz)^\alpha (y^2 z)^{-k}$. The exact ones coming from the homotopies can be written as $(xyz)^{\alpha'+1} (y^2 z)^{-k}$ and hence we see that in this case, there is a single generator corresponding to $\alpha = 0$ which is $(y^2 z)^{-k} = t^{-2k}$.

When $k > 0$, as in the odd case, we have a constraint on α, namely $\alpha \geq 2k$ and the generator corresponding to $\alpha = 0$ no longer survives, hence $\mathrm{Ext}^{2k}(\mathcal{E},\mathcal{E}) = 0, k > 0$. □

We combine these results into a proposition:

Proposition 4.2 (Mirror symmetry for T^*S^2): *There is a derived equivalence*

$$\mathcal{W}(T^*S^2) \simeq D^b(Y^0, x^2z) \simeq D^b(\mathbb{C}[x]/x^2)$$

sending the cotangent fiber which generates the wrapped Fukaya category to the generator of the matrix factorizations

$$F \mapsto x.xz \mapsto \mathbb{C}[x]/x$$

and the zero section which generates the compact Fukaya category to the generator of $\mathrm{Perf}(\mathbb{C}[x]/x)$

$$S^2 \mapsto x^2.z \mapsto \mathbb{C}[x]/x^2$$

In this way, we can see an instance of Koszul duality: the endomorphism ring

$$\mathrm{End}(F) \simeq \mathbb{C}[t]$$

is Koszul dual to the ring corresponding to the zero section, which is $\mathbb{C}[x]/x^2$.

4.3 Mirror symmetry for $K_{\mathbb{P}^1}$

We could compactify in the other direction and introduce a superpotential to the open subset of T^*S^2. We would like to exhibit an equivalence

$$FS((T^*S^2)^0, y) \simeq D^b(K_{\mathbb{P}^1})$$

The derived category of $K_{\mathbb{P}^1}$

We begin with the B-side.

One way to compute the derived category is to use the McKay correspondence: this states that the derived category of the resolution of the A_1-singularity should be the derived category of $\mathbb{Z}_2$-equivariant sheaves on $\mathbb{C}^2$. There are two characters of $\mathbb{Z}_2$ equipping the trivial bundle with two equivariant structures and these generate the derived category. Moreover, these line bundles correspond to $\mathcal{O}, \mathcal{O}(1)$. This can also be done from a VGIT perspective.

We present instead an argument using localisation: we can compactify $K_{\mathbb{P}^1}$ to the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ and use the projective bundle formula:

$$D^b(\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2))) = \langle p^*D^b\mathbb{P}^1 \otimes \mathcal{O}_p(-1), p^*D^b\mathbb{P}^1 \rangle = \langle p^*\mathcal{O}_{\mathbb{P}^1}, p^*\mathcal{O}_{\mathbb{P}^1}(1), p^*\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_p(1), p^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_p(1) \rangle$$

To obtain the derived category of $K_{\mathbb{P}^1}$ we must mod out by the sheaves supported at the divisor at infinity. But notive that there is an ideal sheaf sequence

$$\mathcal{O}_p(-1) \to \mathcal{O} \to \mathcal{O}_\infty$$

When we localize, the relative tautological becomes isomorphic to the structure sheaf and the same is true after tensoring with the line bundle $p^*\mathcal{O}(-1)$

$$\mathcal{O}_p(-1) \simeq \mathcal{O}, \text{ and } \mathcal{O}_p(-1) \otimes p^*\mathcal{O}(-1) \simeq p^*\mathcal{O}(-1) \text{ in } D^b(K_{\mathbb{P}^1})$$

Hence, the generating set of four elements reduces to only $\langle \mathcal{O}, \mathcal{O}(1) \rangle$. Notice, however, that this is no longer a semiorthogonal decomposition, as there are morphisms going both ways.

In fact, the argument presented above works for any line bundle $\mathrm{Tot}(\mathcal{L})$ on any projective space $\mathbb{P}^n$.

We can also identify the quiver algebra defined by these two objects, i.e. the algebra

$$\mathrm{end}(\mathcal{O} \oplus \mathcal{O}(1)).$$

This can be computed by noticing there are no higher Ext groups: this is because $K_{\mathbb{P}^1}$ is a blowup of the affine A_1 singularity. As such, the derived pushforward of the structure sheaf is the structure sheaf: $f_*\mathcal{O}_{K_{\mathbb{P}^1}} = \mathcal{O}$. One can then use adjunction and compute for example that the derived homs behave like they do downstairs: for example,

$$\mathrm{Hom}(\mathcal{O}_{K_{\mathbb{P}^1}}, \mathcal{O}_{K_{\mathbb{P}^1}}) = \mathrm{Hom}(f^*\mathcal{O}, \mathcal{O}_{K_{\mathbb{P}^1}}) = \mathrm{Hom}(\mathcal{O}, f_*\mathcal{O}_{K_{\mathbb{P}^1}}) = H^0(\mathcal{O}) = \mathbb{C}[u,s,t]/(st = u^2)$$

is just the functions on the A_1-singularity, and similarly for $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1))$. Moreover, $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$ is generated by x, y over $\mathrm{Hom}(\mathcal{O}, \mathcal{O})$ and similarly $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O})$ is generated by xz, yz. The resulting algebra is called the *preprojective algebra of extended* A_1.

The Fukaya-Seidel category

To complete the HMS picture, one needs to show that the Lagrangian sections L_0, L_1 generate and that the algebra they generate is quasiequivalent to the preprojective algebra. Unfortunately, we did not have time to confirm this in this project :(

References

[Ab11] Mohammed Abouzaid. *A cotangent fibre generates the Fukaya category*. arXiv:1003.4449 [math]. June 2011. DOI: 10.48550/arXiv.1003.4449. URL: http://arxiv.org/abs/1003.4449 (visited on 06/21/2024).

[AurAb2023] Mohammed Abouzaid and Denis Auroux. *Homological mirror symmetry for hypersurfaces in $(\mathbb{C}^*)^n$*. arXiv:2111.06543 [math]. Jan. 2023. DOI: 10.48550/arXiv.2111.06543. URL: http://arxiv.org/abs/2111.06543 (visited on 05/29/2024).

[AurAb2015] Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov. *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*. arXiv:1205.0053 [math]. July 2015. DOI: 10.48550/arXiv.1205.0053. URL: http://arxiv.org/abs/1205.0053 (visited on 05/29/2024).

[Chan13] Kwokwai Chan. "Homological mirror symmetry for A_n-resolutions as a T-duality". In: *Journal of the London Mathematical Society* 87.1 (Feb. 2013). arXiv:1112.0844 [math], pp. 204–222. ISSN: 00246107. DOI: 10.1112/jlms/jds048. URL: http://arxiv.org/abs/1112.0844 (visited on 06/05/2024).

[ChanUeda13] Kwokwai Chan and Kazushi Ueda. "Dual torus fibrations and homological mirror symmetry for A_n-singularities". In: *Communications in Number Theory and Physics* 7.2 (2013). arXiv:1210.0652 [math], pp. 361–396. ISSN: 19314523, 19314531. DOI: 10.4310/CNTP.2013.v7.n2.a5. URL: http://arxiv.org/abs/1210.0652 (visited on 06/05/2024).

[ChoOh] Cheol-Hyun Cho and Yong-Geun Oh. *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*. arXiv:math/0308225. Aug. 2003. DOI: 10.48550/arXiv.math/0308225. URL: http://arxiv.org/abs/math/0308225 (visited on 06/19/2024).

[Pasc2014] James Pascaleff. "Floer cohomology in the mirror of the projective plane and a binodal cubic curve". In: *Duke Mathematical Journal* 163.13 (Oct. 2014). arXiv:1109.3255 [math]. ISSN: 0012-7094. DOI: 10.1215/00127094-2804892. URL: http://arxiv.org/abs/1109.3255 (visited on 05/29/2024).

[Segal11] Ed Segal. "Equivalences Between GIT Quotients of Landau-Ginzburg B-Models". en. In: *Communications in Mathematical Physics* 304.2 (June 2011), pp. 411–432. ISSN: 0010-3616, 1432-0916. DOI: 10.1007/s00220-011-1232-y. URL: http://link.springer.com/10.1007/s00220-011-1232-y (visited on 06/19/2024).

[Ship2012] Ian Shipman. "A Geometric Approach to Orlov's Theorem". In: *Compositio Mathematica* 148.5 (Sept. 2012). arXiv:1012.5282 [math], pp. 1365–1389. ISSN: 0010-437X, 1570-5846. DOI: 10.1112/S0010437X12000255. URL: http://arxiv.org/abs/1012.5282 (visited on 05/29/2024).

[Vit] C. Viterbo. *Functors and Computations in Floer homology with Applications Part II*. arXiv:1805.01316 [math]. May 2018. doi: 10.48550/arXiv.1805.01316. url: http://arxiv.org/abs/1805.01316 (visited on 06/21/2024).