Cox Rings

More or less: the ring of sections of all effective line bundles.

cf. Cox, The Homogeneous Ring of a Toric Variety

Let $X = TV(\Sigma)$, with $\Sigma \subset N$. Let u_ρ be the primitive generator of the the ray ρ, corresponding to T-invariant divisor D_ρ. Assume that Σ spans $N_\mathbb{R}$, that is, X has no torus factors. Let $\Sigma(1)$ (or $\sigma(1)$) be the set of 1-dimensional faces of a fan (or cone).

Let $\mathbb{Z}^{\Sigma(1)} = \bigoplus_\rho \mathbb{Z} \cdot D_\rho$

Define the cox ring $S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$

A monomial $\prod_\rho x_\rho^{a_\rho}$ determines the divisor $D = \sum a_\rho D_\rho =: x^D$

Define the degree of a monomial $deg(x^D) = [D] \in A_{n-1}(X)$. Then two monomials x^a, x^b are of the same degree iff there exists a $m \in M$ such that $a_\rho = \langle m, n_\rho \rangle + b_\rho$ for all ρ. Define the grading by

$$S_a = \oplus_{deg(x^D)=a} \mathbb{C} \cdot x^D$$

so we can write S as

$$S = \bigoplus_{a \in A_{n-1}(X)} S_a$$

and $S_a \cdot S_b \subset S_{a+b}$, which is the homogeneous coordinate ring with respect to the grading.

e.g.
X = $\mathbb{P}^n$, S = $\mathbb{C}[x_0, \ldots, x_n]$
X = $\mathbb{P}(a_0, \ldots, a_n)$, S = $\mathbb{C}[x_0, \ldots, x_n]$ with each x_i given weight a_i
X = $\mathbb{P}^n \times \mathbb{P}^m$, then $S = \mathbb{C}\left[x_0, \ldots, x_n; y_0, \ldots, y_m\right]$. Here, the grading is the usual

bigrading, where a polynomial has bidegree (a, b) if it is homogeneous of degree a (resp. b) in the x_i (resp. y_j).

Proposition: If $\alpha=[D] \in A_{n-1}(X)$, then there is an isomorphism

$$\phi_D : S_\alpha \simeq H^0\left(X, \mathcal{O}_X(D)\right)$$

where $\mathcal{O}_X(D)$ is the coherent sheaf on X determined by the Weil divisor D (see Fulton, §3.4). If $\alpha=[D]$ and $\beta=[E]$, then there is a commutative diagram

$$\begin{array}{ccc} S_\alpha \otimes S_\beta & \longrightarrow & S_{\alpha+\beta} \\ \downarrow & & \downarrow \\ H^0\left(X, \mathcal{O}_X(D)\right) \otimes H^0\left(X, \mathcal{O}_X(E)\right) & \longrightarrow & H^0\left(X, \mathcal{O}_X(D+E)\right) \end{array}$$

where the top arrows is multiplication, the bottom arrow is tensor product, and the vertical arrows are the isomorphisms $\phi_D \otimes \phi_E$ and ϕ_{D+E}.

This comes from the fact that $H^0(X, \mathcal{O}_X(D))=\oplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$

For X a complete toric variety, S_a is finite-dimensional for all a, and $S_0=\mathbb{C}$. Moreover, if $a=[D]$ for an effective divisor D, then $dim_{\mathbb{C}} S_a=\mid P_D \cap M \mid$

WARNING: This ring developed so far depends only on the one-dimensional cones of the fan, but not higher-dimensional cones.

Define the ideal B_X of S (which depends on the whole fan) as follows: For each cone $\sigma \in \Sigma$, Let $\hat{\sigma}=\sum_{\rho \notin \sigma(1)} D_\rho$, and define $x^{\hat{\sigma}}=\prod_{\rho \notin \sigma(1)} x_\rho$. Then $B:=\left\langle x^{\hat{\sigma}} : \sigma \in \Sigma\right\rangle$

Note the B is in fact generated by the $x^{\hat{\sigma}}$ as σ ranges over the maximal cones of Σ. Also, if Σ and Σ' are fans in N with $\Sigma(1)=\Sigma^{'}(1)$, then $\Sigma=\Sigma'$ if and only if the corresponding ideals $B, B' \subset S$ satisfy $B=B'$. Because $\left\{x^{\hat{\sigma}}\right.$: σ is a maximal cone of $\left.\Sigma\right\}$ = minimal basis of the monomial ideal B.

When X is a projective space or weighted projective space, the ideal B is $\langle x_0, \ldots, x_n\rangle \subset \mathbb{C}\left[x_0, \ldots, x_n\right]$. $B \subset S$ plays the role for an arbitrary toric variety that $\langle x_0, \ldots, x_n\rangle \subset \mathbb{C}\left[x_0, \ldots, x_n\right]$ plays for projective space. In particular, we have the variety

$$Z=\mathbf{V}(B)=\left\{x\in\mathbb{C}^{\Sigma(1)}: x^{\hat{\sigma}}=0\text{ for all }\sigma\in\Sigma\right\}\subset\mathbb{C}^{\Sigma(1)}$$

as the irrelevant subset of $\mathbb{C}^{\Sigma(1)}$.

Lemma: $Z\subset\mathbb{C}^{\Sigma(1)}$ has codimension at least two.
Proof: Since $B'=\left\langle x^{\hat{\rho}}:\rho\in\Delta(1)\right\rangle\subset B$, we have $Z=\mathbf{V}(B)\subset\mathbf{V}\left(B'\right)$. But $\mathbf{V}\left(B'\right)$ is the union of all codimension two coordinate subspaces, and the lemma follows.

Now: how does this define a quotient?

Let X be toric variety arising from the fan $\Sigma\subset N$, with $M=\operatorname{Hom}(N,\mathbb{Z})$. There is an exact sequence

$$M\to\operatorname{Div}_{T_N}(X)=\bigoplus_\rho\mathbb{Z}D_\rho\to Cl(X)\to 0$$

sending a character $\chi^m\mapsto\sum\operatorname{div}(\chi^m)\left\langle u_\rho,m\right\rangle D_\rho$ and the second map sending a divisor to its class in ClX). Moreover, the u_ρ's span $N_{\mathbb{R}}$, if and only if we have the exact sequence $$
0\to M \to \mathrm{Div}{T{N}}(X)\to Cl(X) \to 0

$Assumethat\$X\$hasnotorusfactors, i.e. thesecondexactsequenceholds. Thenapplying\$$

Let $G:=\operatorname{Hom}(\operatorname{Cl}(X),\mathbb{C}^*)$ as in the above sequence. Then some properties of G are:

- $Cl(X)$ is the character group $\hat{G}$ of G
- $G\simeq(\mathbb{C}^*)^l\times H$, with H finite, which comes from the fact that $Cl(X)$ is a finitely generated abelian group of rank $l=d-n$, where $d=|\Sigma(1)|$

Since $\left(\mathbb{C}^*\right)^{\Sigma(1)}$ acts naturally on $\mathbb{C}^{\Sigma(1)}$, the subgroup $G\subset\left(\mathbb{C}^*\right)^{\Sigma(1)}$ acts on $\mathbb{C}^{\Sigma(1)}$ by

$$g\cdot t=\left(g\left([D_\rho]\right)t_\rho\right)$$

for $g: A_{n-1}(X)\to\mathbb{C}^*$ in G and $t=(t_\rho)$ in $\mathbb{C}^{\Sigma(1)}$.

Using the action of G on $\mathbb{C}^{\Delta(1)}$, we can describe the toric variety X as the quotient $X = \mathbb{C}^{\Sigma(1)} - Z//G$.

Construction from weights:

Let's recall some basics of toric geometry. Let $M = \mathrm{Hom}(T^n, \mathbb{C}^*)$, and $N = M^\vee$. Recall that toric varieties are determined by their fan in $N_\mathbb{R}$, with an exact sequence and its dual (assume no torus factors)

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\rho} N \to 0$$

and

$$0 \to M \to (\mathbb{Z}^m)^\vee \xrightarrow{Q} \mathbb{L}^\vee \to 0$$

The map Q describes an action of $(\mathbb{C}^*)^n$ on the vector space $\mathbb{C}^m$, given by a $n \times m$ weight matrix, so we can form a GIT quotient with respect to this action. The anticanonical divisor (here denoted $\det V$) is associated to the sum $\det V = \sum_{q_i \in Q} q_i$ of the column of the weight matrix. Moreover, we call a Toric GIT problem Calabi-Yau if $\det V = 0$.

We can define semi stable loci by a choice of character χ in $\mathbb{L}^\vee$ (which corresponds to a $(\mathbb{C}^*)^n$ -linearised line bundle L_χ on $\mathbb{C}^m$).

$$X^{ss}(L_\chi) = \{a \in \mathbb{C}^m : \quad \exists\, n > 0, f \in \Gamma(L_{n\chi}) \text{ s.t. } f(a) \neq 0\}$$

In practice, the semistable locus for a certain character $X^{ss}(L_\chi)$ is calculated as the complement of the vanishing locus of the irrelevent ideal.

For a stability condition χ in the secondary fan, define the irrelevant ideal Irr_χ as

$$Irr_\chi = (x_{i_1}, \ldots, x_{i_r} \mid \chi \in \langle q_{i_1}, \ldots, q_{i_r} \rangle_+)$$

That is, the ideal generated by monomials corresponding to cones containing χ. Then $X^{us}(L_\chi) = V(Irr)$

From this we get the quotient $\mathbb{C}^m //_\chi T^n := (\mathbb{C}^m - X^{us}(L_\chi))/T^n$

The columns of the weight matrix generate rays of a fan in $\mathbb{L}^\vee$, which we call the secondary fan. This gives a wall and chamber decomposition of characters. It can be shown that two stability conditions chosen from the interior of the same chamber will give the same quotient.

Example:
Consider the action of $(\mathbb{C}^*)^2 = T^2$ on $V = \mathbb{C}^4$, with weight matrix

$$Q = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and action $(\lambda, \mu)(x_1, x_2, x_3, x_{4)} = \left(\lambda x_1, \lambda x_2, \mu x_3, \frac{\mu}{\lambda^2} x_4\right)$

Define $\det V$ as the sum of the columns in the weight matrix, in this case $(0, 2)^T$
The characters define GIT quotients:

- $X_1 = \mathbb{C}^4 //_{\chi_1} T^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) = \mathbb{F}_2$
- $X_2 = \mathbb{C}^4 //_{\chi_2} T^2 = \mathbb{P}(1, 1, 2)$

Note that $\mathbb{F}_2$ is the minimal resolution of $\mathbb{P}(1, 1, 2)$, related by a blow up at its singular point, but as orbifolds, they are related by a flop.

Wall crossings can give us other standard birational transformations

example:
Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^3$ with weights $(1, 1, -1)$.
There are two quotients: X_+ corresponding to the chamber with the weights $(1, 1)$, i.e. take the unstable locus to be $x = y = 0$, or X_- corresponding to the chamber with -1, i.e. take unstable locus $z = 0$. With these stability conditions we have

$$X_+ = \mathcal{O}(-1)_{\mathbb{P}^1} \qquad X_- = \mathbb{C}^2,$$

where the wall crossing from the X_- to X_+ gives the blow up at a point.

Now suppose $\mathbb{C}^*$ now acts on $V = \mathbb{C}^4$ with coordinates x_1, x_2, y_1, y_2, and weight matrix $Q = (1 \quad 1 \quad -1 \quad -1)$.
Defines two chambers in the secondary fan: $\chi_+ > 0$ and $\chi_- < 0$, so we get

unstable locus $x_1 = x_2 = 0$ and $y_1 = y_2 = 0$. Hence

$$X_+ \simeq \mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1} \simeq X_-$$

This is an example of the Atiyah flop, related by a blow up and its flopping contraction.

In the MMP, we say a variety is a minimal model if has nef canonical divisor. In the GIT picture, we say a GIT quotient is minimal if $-\det V$ lies in the closure of the chamber corresponding to the variety -- The character defining the semistable locus lies in the nef cone

Wall crossing Formula

Let V be a vector space of dimension n, and let T be an algebraic torus. Denote $\det V = \sum_i q_i$, where q_i is the i^{th} column of the weight matrix Q.

Consider a toric GIT problem defined by the action of a group T on a vector space V. Let C_+ and C_- be adjacent chambers of the secondary fan in $L^*_{\mathbb{R}}$ separated by a wall W. Assume that $\det V$ is on the C_+ side of the adjoining wall W. The wall W corresponds to an orthogonal (primitive) one-parameter subgroup $\lambda_W \in L$.
We can define a value $\kappa = (\det V)(\lambda_W)$. Let λ_W be such that $\kappa \geq 0$, so is pointing to the C_+ 'side' of the wall. κ is a combinatorial value which will (roughly) tell us which chamber admits the 'bigger' GIT quotients.

Let X_+ (resp. X_-) be the GIT quotient $V//_{\theta_+}T$ (resp. $V//_{\theta_-}T$) corresponding to the chosen generic stability condition $\theta_+ \in C_+$ (resp. θ_-). Recall from the previous section that GIT quotients are invariant across stability conditions in the interior of a given chamber.

We can define a somewhat 'smaller' GIT problem associated to a subset $S \subset \{1, \ldots, n\}$, or more specifically a subset Q_S of the weights corresponding to the set S, which in our case are the s_i columns of the weight matrix for $s_i \in S$. These weights generate a sublattice $L^*_S \subset L^*_{\mathbb{R}}$, determining what we call a Higgs

GIT problem, defined by the exact sequence

$$M_S \to \mathbb{Z}^S \xrightarrow{Q_S} L_S^*$$

From this we will now form a strictly lower dimensional variety Z which give components in a SOD of X_+. First, we form a Higgs GIT problem, which defines the GIT quotient of the fixed locus V^{λ_W} by T/λ_W. Here, our subset Q_S is the collection of weights which are orthogonal to λ_W, that is, the weights which lie in the space spanned by W. We can see that the lattice L_S^* is exactly the character lattice for the action of T/λ_W, since the the weights span the space orthogonal to λ_W. Moreover, the subspace of V fixed by λ_W corresponds to the lattice $\mathbb{Z}^S$ in the exact sequence. We choose a character θ_W in the chamber of L_S^* corresponding to the cone generated by W, to form the quotient

$$Z = V^{\lambda_W} //_{\theta_W} (T/\lambda_W).$$

Hence we get the theorem due to HL and BFK.

Theorem 1.

Consider GIT quotients X_+, X_- related by a wall crossing across the wall W as described above.
If $\kappa > 0$, we have a semi-orthogonal decomposition given by

$$D(X_+) = \langle D(X_-), D(Z), \ldots, D(Z) \rangle$$

with κ copies of $D(Z)$ appearing.
If $\kappa = 0$, the wall crossing induces a flop, and we have an equivalence of categories

$$D(X_+) \simeq D(X_-).$$

This theorem was proved in BFK in much greater generality than used here, where such a decomposition holds for a smooth quasi-projective variety acted upon by a linear algebraic group, and a wall-crossing between two G-equivariant line bundles. However, to state the theorem in full generality requires more technical machinery than is necessary for the toric case for the purposes of our examples below.

For ease of notation, we will denote the factor of $D(X)$ in a semi-orthogonal decomposition just as X.

Beilinson's collection

Recall Beilinson's exceptional collection which forms a SOD of $D(\mathbb{P}^n)$. Since $\mathbb{P}^n$ is a toric variety, we can realise it as the GIT quotient with respect to the usual action of $\mathbb{C}^*$ on $V = \mathbb{C}^{n+1}$. So the weights are $(1 \quad 1 \quad \dots \quad 1)$, with $\det V = n + 1$. The wall crossing to $X_- = \emptyset$, retrieves the decomposition $\langle pt, \dots, pt \rangle$ with $n + 1$ copies of the derived category of a point, corresponding to the exceptional collection of line bundles on $\mathbb{P}^n$.

Blow Up

Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^3$ with weights $(1, 1, -1)$. There are two stability conditions which can define toric GIT quotients, with X_+ corresponding to the chamber with the weights $(1, 1)$, i.e. take the unstable locus to be $x = y = 0$, or X_- corresponding to the chamber with -1, i.e. take unstable locus $z = 0$. With these stability conditions we have

$$X_+ = \mathcal{O}(-1)_{\mathbb{P}^1} \qquad X_- = \mathbb{C}^2,$$

Hence we get the have the decomposition

$$D(\mathcal{O}(-1)_{\mathbb{P}^1}) = \langle \mathbb{C}^2, pt \rangle$$

which recovers Orlov's blow-up formula, as X_+ is the blow-up of $\mathbb{C}^2$ at a point.