Cubic fourfolds and mirror symmetry

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Overview

- Cubic fourfolds, their moduli space and the rationality problem
- Hodge theoretic vs derived categorical perspective
- Example: the class of cubics containing a plane
- Addington-Thomas: equivalence between the Hodge theoretic and derived conjectures
- Sheridan-Smith and HMS for cubic fourfolds

Rationality of cubic fourfolds

We know classically that cubic curves are not rational (they are elliptic curves). However, cubic surfaces are always rational! Clemens and Griffiths showed in 1972 that cubic threefolds are, in contrast, never rational.

What about cubic fourfolds? There is some locus of cubic fourfolds which are rational, but it is (conjecturally) neither open nor closed! Let's begin to understand their moduli space via the Torelli theorem of Voisin:

Theorem (Voisin): *The period map*

$$\mathcal{C} \to \mathcal{D}$$

sending a cubic fourfold X *to* $\{H^{1,3}(X) \subset H^4(X, \mathbb{C})\}$ *is an open immersion.*

Hodge theory cubic fourfolds

Hodge theory of the cubic fourfold

Let $X \subset \mathbb{P}^5$ be the zero locus of a degree 3 polynomial. We can compare its Hodge diamond to that of a K3 surface:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Cubic fourfold

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

K3 surface

If we take the primitive cohomology of the cubic fourfold, we get $(0, 1, 20, 1, 0)$ which looks exactly like the middle cohomology of a K3 surface!

Cubic fourfolds and K3 surfaces

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&0&&0&&0& \\ 0&&1&&21&&1&&0 \\ &0&&0&&0&&0& \\ &&0&&1&&0&& \\ &&&0&&0&&& \\ &&&&1&&&& \end{array}$$

Cubic fourfold

$$\begin{array}{ccccc} &&1&& \\ &0&&0& \\ 1&&20&&1 \\ &0&&0& \\ &&1&& \end{array}$$

K3 surface

However, the two have different signatures. To be able to compare them, we have to pass to a codimension one subspace. For the K3, we can always take the primitive cohomology.

For *special cubic fourfolds* we can find a class $T \in H^{2,2}(X, \mathbb{Z})$ and move to its orthogonal complement (a generic cubic has no such class!)

Hasset's theorem

Because of the Torelli theorem, finding the class of cubic fourfolds for which such a surface exists and moreover has an associated K3 surface is a lattice-theoretic problem:

Theorem (Hasset): *The cubics containing an integral class T as above form a family of irreducible divisors $\mathcal{C}_d$, nonempty iff (*) $d > 6, d = 0, 2 \pmod 6$. Moreover, the cubics in $\mathcal{C}_d$ have an associated K3 surface i.e. $\exists S$ such that*

$$H^2_{prim}(S;\mathbb{Z})(-1) \simeq \langle h^2, T\rangle^\perp$$

precisely when (**) d is not divisible by four, nine, or any odd prime $p = -1 (\bmod 3)$

Derived perspective

The derived perspective

Kuznetsov presented a different viewpoint on the cubic fourfold using derived categories. The three line bundles $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$ form an exceptional collection and he defined the component

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp} \subset \mathcal{D}(X)$$

Kuznetsov showed that $\mathcal{A}_X$ is a *Calabi-Yau 2* category and posed the conjecture:

Conjecture (Kuznetsov): *X is rational if and only if there is a K3 surface S such that*

$$\mathcal{D}(S) \simeq \mathcal{A}_X$$

Cubics containing a plane and twisted K3 surfaces

Fundamental (non)-example: cubics containing a plane

The first nonempty item on Hasset's list consists of the cubics containing a plane. This is $\mathcal{C}_8$ (8 is coming from the intersection matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$).

This class of cubics is perhaps the most important one: Voisin used it to prove the Torelli theorem for cubic fourfolds, and Addington-Thomas showed that $\mathcal{C}_8 \cap \mathcal{C}_d \neq \emptyset$ for all other nonempty $\mathcal{C}_d$ and used deformation theory out of $\mathcal{C}_8$ to deduce that Hasset's and Kuznetsov's conjectures are in fact equivalent! Moreover, we can see how the associated K3 surface appears geometrically.

Cubics containing a plane

If $P \subset X \subset \mathbb{P}^5$ is a cubic fourfold containg a plane, then there is a residual quadric fibration More on the geometry

$$\mathrm{Bl}_P X = \tilde{X} \to \mathbb{P}^2 = \{\text{3-planes in } \mathbb{P}^5 \text{ containing } P\}$$

The fibers are generically $\mathbb{P}^1 \times \mathbb{P}^1$ degenerating to a cone over a sextic curve in the base. Now, $\mathbb{P}^1 \times \mathbb{P}^1$ has exactly two rulings, parametrized by the connected components of the Fano variety of lines

$$\widetilde{\mathscr{F}}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}^1 \coprod \mathbb{P}^1$$

A singular quadric is a cone, so has exactly one ruling:

$$\widetilde{\mathscr{F}}(Q) = \mathbb{P}^1$$

If we define S to be the space of rulings on the fibers, we see that it is a double cover of $\mathbb{P}^2$ branched along the sextic curve. Moreover, the relative Fano variety parametrizing lines in the fibers is a $\mathbb{P}^1$ bundle over this:

$$\widetilde{\mathscr{F}} \xrightarrow{\pi_S} S \xrightarrow{\pi} \mathbb{P}^2$$

Twisted K3 surfaces

In fact, this S is a K3 surface. There is an obstruction cocycle α to $\widetilde{\mathscr{F}}$ being a projectivization of a vector bundle, governed by the exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL} \to \mathrm{PGL} \to 1$$

We can consider *twisted vector bundles* on S which only satisfy the cocycle condition up to α, and they have a derived category $\mathcal{D}(S, \alpha)$.

Theorem (Bernardara's twisted projective bundle formula):
There is a semiorthogonal decomposition

$$\mathcal{D}(\widetilde{\mathscr{F}}) = \langle \mathcal{D}(S, \alpha), \mathcal{D}(S) \rangle$$

Kuznetsov's equivalence: mutation

Kuznetsov's original proof is to compare two different SOD's for $\tilde{X}$: one coming from its realization as a quadric fibration:

$$\mathcal{D}(\tilde{X}) \simeq \langle \mathcal{D}(S,\alpha), \mathcal{D}(\mathbb{P}^2), \mathcal{D}(\mathbb{P}^2) \otimes \mathcal{O}(H) \rangle \simeq$$
$$\simeq \langle \mathcal{D}(S,\alpha), \mathcal{O}(-h), \mathcal{O}, \mathcal{O}(h), \mathcal{O}(H), \mathcal{O}(h+H), \mathcal{O}(2h+H) \rangle$$

the other as a blowup:

$$\mathcal{D}(\tilde{X}) \simeq \langle \mathcal{D}(X), \mathcal{D}(\mathbb{P}^2) \rangle \simeq$$
$$\simeq \langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H), \mathcal{O}_E, \mathcal{O}_E(H), \mathcal{O}_E(2H) \rangle$$

Via a sequence of mutations, one is transported into the other, identifying the twisted category $\mathcal{D}(S,\alpha)$ with the Kuznetsov component.

Kuznetsov's equivalence: FM kernel

Theorem (Kuznetsov): *There is a derived equivalence*

$$\mathcal{A}_X \simeq \mathcal{D}(S, \alpha)$$

One way to do this is via an FM kernel from the universal line $\widetilde{\mathscr{P}} \subset X \times \widetilde{\mathscr{F}}$ with ideal sheaf $\mathcal{I}$ and its adjoint:

$$\begin{array}{ccc} \mathcal{D}(X) & \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} & \mathcal{D}(\widetilde{\mathscr{F}}) \\ \uparrow & & \uparrow \\ \mathcal{A}_X & \underset{\simeq}{\longleftarrow} & \mathcal{D}(S, \alpha) \end{array}$$

Relation to other cubic fourfolds

We have just seen that the cubic fourfolds with a plane behave like twisted K3 surfaces. A natural question to ask is when are they actually geometric, i.e. $\alpha = 1$?

Theorem (Kuznetsov): *The Brauer class vanishes precisely when there is another surface class W satisfying $\deg(W) - P \cdot W$ being odd. This is equivalent to $X \in \mathcal{C}_8 \cap \mathcal{C}_d$ where C_d is one of the divisors with associated K3 surfaces.*

Addington-Thomas' argument

Addington-Thomas show that Hassett and Kuznetsov's conditions are equivalent.

First, Hassett's arithmetic condition of special cubics is replaced by the following:

$$X \text{ is special} \iff \exists \kappa_1, \kappa_2 \in K_{alg}(\mathcal{A}_X):$$
$$\chi(\kappa_1, \kappa_1) = 0, \chi(\kappa_1, \kappa_2) = 1, \chi(\kappa_2, \kappa_2) = 2$$

Morally, $\kappa_1 = \mathcal{O}_p$ and $\kappa_2 = \mathcal{O}_S$ on the associated K3 surface.

We understand the case of $X \in \mathcal{C}_8 \cap \mathcal{C}_d$, where there is an FM equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{D}(S)$. Addington-Thomas then deform the kernel as follows:

- Consider the first-order deformations of X, S and Φ: these controlled by the action of Φ on cohomology and in particular by the Kodaira-Spencer classes κ_S, κ_X. By modifying the equivalence Φ appropropriately, this can be shown to vanish.
- Extend the first-order deformations to all orders, by using T^1-lifting methods.

Mirror symmetry for cubic fourfolds

Sheridan-Smith: B-side

We consider the degenerating family of cubic fourfolds

$$Z_b = \{f_b = 0\} \subset \mathbb{P}^5_\Lambda$$
$$f_b = -z_1 z_2 z_3 - z_4 z_5 z_6 + \sum_{p \in \Delta} b_p \mathbf{z}^{\mathbf{p}}$$

The p's in the sum correspond to the set Δ consisting of tuples $(p_1, p_2, p_3, p_4, p_5, p_6) \in \mathbb{Z}^6_{\geqslant} 0$ with $\sum p_i = 3$ and at most one of (p_1, p_2, p_3) being non-zero and the same for (p_4, p_5, p_6). In fact, $|\Delta| = 24$ and the monomial terms correspond to the components of the boundary divisor D of the mirror K3 surface, under the monomial-divisor mirror map.

When $b = 0$, we get the most singular cubic fourfold, the Perazzo primal.

Sheridan-Smith: A-side

We take a K3 surface S which is the resolution of the quotient $(E \times E)/\mathbb{Z}_3$ where $E = \mathbb{C}/\langle 1, e^{2\pi i/6} \rangle$ is an elliptic curve. This has a divisor D with 24 components and we consider the family of Kahler forms $\omega_\lambda = \sum \lambda_p PD[D_p]$. These classes span a 20-dimensional space in $H^2(S)$.

Thinking in the relative Fukaya category $\mathcal{F}(S, D)$, sending $\lambda_p \to \infty$ corresponds to removing a component D_p of the divisor. At the limit where all $\lambda_p \to \infty$, we get $S \setminus D$, the mirror to the Perazzo primal.

Sheridan-Smith: HMS for the Kuznetsov component

Theorem (Sheridan-Smith): *There is a quasiequivalence*

$$\mathcal{F}(S, \omega_\lambda) \simeq \mathrm{GrMF}(f_{b(\lambda)}) \simeq \mathcal{A}_{Z_b}$$

where $\mathrm{val}(b_p) = \lambda_p$.

The idea is to verify this at the central fiber, when we get the most singular cubic fourfold, but the least complicated A_∞ structure, and then deform out, by using a versality argument.

In particular, combining this with the Addington-Thomas criterion, we might expect that a cubic fourfold Z is rational if and only if the mirror K3 surface admits an SYZ torus fibration with a Lagrangian section!

HMS for the cubic fourfold

More generally, one can apply a variant of the AAK construction to the cubic fourfold to extend this to the whole of $\mathcal{D}(Z)$.

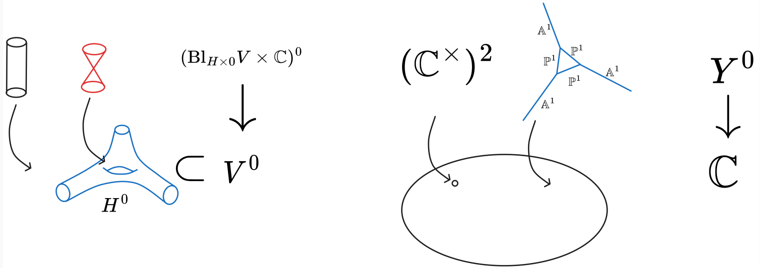

Figure 1: AAK construction

HMS for the cubic fourfold

The mirror to our degenerating family of cubic hypersurfaces Z_b is a toric 6-fold Y, equipped with a superpotential

$$W = \underbrace{\sum_i z^{e_i^*}}_{\text{from } \mathbb{P}^5} + \underbrace{z^{\frac{1}{3}\sum e_i^*}}_{\substack{\text{vanishes to order 1} \\ \text{along the boundary divisor}}}$$

In more detail, we have chosen

- A lattice $L = \{\mathbf{v} \in \mathbb{Z}^6 | \sum v_i = 0 (\mod 3)\}$
- A simplex $\Delta = \{\mathbf{v} \in \mathbb{Z}^6_{\geqslant 0} | \sum v_i = 3\}$
- A function $\lambda : \Delta \to \mathbb{R}_{\geqslant 0}$ with $\lambda^{-1}(0) = \{e_1 + e_2 + e_3, e_4 + e_5 + e_6\}$ and a convex extension ψ_λ. This gives us the family

$$Z_\lambda = \{\sum b_{\mathbf{v}} \mathbf{z}^{\mathbf{v}} = 0\}, b_{\mathbf{v}} = t^{\lambda(v)} + h.o.t$$

- A fan Σ_λ given by the maximal domains of linearity of ψ_λ corresponding to the mirror toric variety Y_λ.

HMS for the cubic fourfold

We can reduce Y_λ to something simpler, namely

$$Y_\lambda = \mathrm{Tot}\left(\mathcal{L}_1 \oplus \mathcal{L}_2\right) \text{ over a toric 4-fold } \overline{Y}_\lambda$$

$$\overline{Y}_\lambda \leftrightarrow \Sigma_\lambda/(e_1 + e_2 + e_3, e_4 + e_5 + e_6)$$

$$D_1 = \mathcal{L}_1 \oplus \{0\} \leftrightarrow e_1 + e_2 + e_3$$

$$D_2 = \{0\} \oplus \mathcal{L}_2 \leftrightarrow e_4 + e_5 + e_6$$

The terms of the superpotential have the following orders of vanishing:

Terms of W	Order along D_1	Order along D_2
$z^{e_1^*}, z^{e_2^*}, z^{e_3^*}$	1	0
$z^{e_4^*}, z^{e_5^*}, z^{e_6^*}$	0	1
$z^{\frac{1}{3}\sum e_i^*}$	1	1

So along $D_1 \cap D_2 = \overline{Y}_\lambda$, locally we can identify Y_λ with $\overline{Y}_\lambda \times \mathbb{C}^2_{x,y}$ and

$$W = \underbrace{xg_1(z) + yg_2(z)}_{W_0} + \underbrace{xyh(z)}_{W_1}$$

HMS for the cubic fourfold

Now, the claim is that

$$FS_0(W) \simeq \mathcal{F}(g_1^{-1}(0) \cap g_2^{-1}(0)) \simeq \mathcal{F}(S)$$

where $S = \{z^{e_1^*} + z^{e_2^*} + z^{e_3^*} = 0\} \cap \{z^{e_4^*} + z^{e_5^*} + z^{e_6^*} = 0\} \subset \overline{Y}_\lambda$ is the K3 surface from Sheridan-Smith. Finally, when we add in the term W_1, this adds in three Morse singularities and

$$FS(W_0 + W_1) \simeq \langle \mathcal{F}(S), L_1, L_2, L_3 \rangle \simeq \mathcal{D}(Z)$$

Thank you for your attention!

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Cubics containing a plane: geometry

Cubics containing a plane: geometry

Projecting orthogonally out of a plane $P \subset \mathbb{P}^5$ gives a rational map

$$\begin{array}{ccccc}
\mathbb{P}\mathcal{N}_{P/\mathbb{P}^5} \simeq E & \longrightarrow & \mathrm{Bl}_P \mathbb{P}^5 & & \\
\downarrow & & \downarrow \tau & \searrow^{\phi} & \\
P & \longrightarrow & \mathbb{P}^5 & \dashrightarrow & \mathbb{P}^2
\end{array}$$

In coordinates, if $P = \mathbf{V}(z_0, z_1, z_2)$, then the rational map is given by the linear system $|\mathcal{I}_P \otimes \mathcal{O}(1)|$ which has basis z_0, z_1, z_2 and base locus P. It is resolved by blowing up, and the map ϕ is given by the linear system $\mathcal{I}_E \otimes \tau^*\mathcal{O}(1)$.

Hence, we see that

$$\mathcal{I}_E \otimes \tau^*\mathcal{O}(1) \simeq \phi^*\mathcal{O}(1)$$

If we denote by H the hyperplane class in $\mathbb{P}^5$ and by h the one on $\mathbb{P}^2$, then this implies that

$$E \sim H - h$$

Cubics containing a plane: geometry

We can think of the blowup in this simplified setting as the set

$$\mathrm{Bl}_P\mathbb{P}^5 = \{p \in \mathbb{P}^5, q \in \mathbb{P}^2 \,|\, p_i q_j = p_j q_i, 0 \leqslant i, j \leqslant 2\} \subset \mathbb{P}^5 \times \mathbb{P}^2$$

which comes equipped with two projection maps to $\mathbb{P}^5$ and $\mathbb{P}^2$ which are the blowup and resolution maps, respectively. We can clearly see that over $\mathbb{P}^2$, this is a projective bundle by looking at the affine cone over $\mathbb{P}^5$:

$$\{a \in \mathbb{A}^6, q \in \mathbb{P}^2 \,|\, a_i q_j = a_j q_i, 0 \leqslant i, j \leqslant 2\} \to \mathbb{P}^2$$

We can see that the coordinates a_3, a_4, a_5 are unconstrained, so give us a copy of the trivial bundle $\mathcal{O}^{\oplus 3}$. The other three coordinates give us precisely the tautological bundle:

$$\mathcal{O}(-1) = \{(a_0, a_1, a_2) \in \mathbb{A}^3, q \in \mathbb{P}^2 \,|\, (a_0, a_1, a_1) \in q\}$$

Hence, we conclude that

$$\mathrm{Bl}_P\mathbb{P}^5 = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 3}) := \mathbb{P}(\mathcal{F}^\vee)$$

Cubics containing a plane: geometry

The strict transform has $\tilde{X} \sim 3H - E \sim 2H + h$ and is thus a quadric fibration

$$\tilde{X} \to \mathbb{P}^2$$

which geometrically can be thought of as follows: the $\mathbb{P}^2$ parametrizes 3-planes containing P, and X intersects such a 3-plane in P union a quadric.

We can see $\tilde{X}$ as the zero locus of $q \in H^0(\mathbb{P}^2, S^2\mathcal{F} \otimes \mathcal{O}(1))$ which can also be thought of as a map $\mathcal{F} \to \mathcal{F}^* \otimes \mathcal{O}(1)$. The quadric fibers are nonsingular outside the determinant locus, which is a sextic curve, since $\det(q) = \mathcal{O}(6)$. back