Exceptional collections & Kiup's conjecture

(Recall)

Notation: $\mathcal{D}$ is derived category (can be any, but one rather should think of $\mathcal{D}^b(X)$).

$\mathrm{Hom}^i(X,Y)$ is the same as $\mathrm{Ext}^i(X,Y)$.

__Def__: $E \in \mathcal{D}$ is __exceptional__ if

base field

$$\mathrm{Hom}^i(E,E) = \begin{cases} k & \text{if } i=0 \\ 0 & \text{otherwise.} \end{cases}$$

__Def__: Let $E_1, \ldots E_n$ be a sequence of exceptional objects, we call it __exceptional sequence__ if there are no Homs from left to right, i.e.,

$$\mathrm{Hom}^k(E_i, E_j) = 0 \quad \forall k;\ i > j.$$

i.e. they generate the whole $\mathcal{D}$

We call it __full__ if $\langle E_1, \ldots E_n \rangle = \mathcal{D}$.

Why do we care? $\rightsquigarrow$ it gives a semi-orthogonal decomposition.

__Obs__: If E is exceptional $\Rightarrow$ $\langle E \rangle$ is admissible.

consists of elmts of the form $\oplus E[i]^{\oplus}$

[Recall: $\mathcal{D}' \hookrightarrow \mathcal{D}$ is __admissible__ if $\forall A \in \mathcal{D}$ we have

$$\underset{\in \mathcal{D}'}{D} \longrightarrow A \to \underset{\in (\mathcal{D}')^\perp}{D'} \to D[1] \text{ dist}$$

i.e. we can decompose somehow].

Idea: $\exists$ canonical evaluation morphism

$$\underbrace{\bigoplus \mathrm{Hom}(E, A[i]) \otimes E[-i]}_{\in \langle E \rangle} \to A \to B$$

cone of the map

(one can easy check that $B \in (\langle E \rangle)^{\perp}$).

<u>Def</u>: a sequence of full admissible tr. subcateg.

$$\mathcal{D}_1 \text{ — } \mathcal{D}_n \subset \mathcal{D}$$

is semi-orthogonal if $\mathcal{D}_j \subset \mathcal{D}_i^{\perp}$ for $i > j$.

in other words we have $\mathrm{Hom}(\mathcal{D}_i, \mathcal{D}_j) = 0$

Moreover, it gives a filtration for every object of $\mathcal{D}$.

Namely, for any $A \in \mathcal{D}$ we can do the following

$$\underbrace{A_1}_{\in \langle \mathcal{D}_2 - \mathcal{D}_n \rangle} \to A \to \underbrace{\mathcal{E}_1}_{\in \langle \mathcal{D}_2 - \mathcal{D}_n \rangle^{\perp} = \mathcal{D}_1} \to A_1[1]$$

continuing this process we get the filtration

$$\begin{tikzcd}
0 = A_n \arrow[rr] & & A_{n-1} \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] & & A_1 \arrow[rr] \arrow[dl] & & A = A_0 \arrow[dl] \\
& \mathcal{E}_n \in \mathcal{D}_n \arrow[ul, dashed] & & & & \mathcal{E}_2 \in \mathcal{D}_2 \arrow[ul, dashed] & & \mathcal{E}_1 \in \mathcal{D}_1 \arrow[ul, dashed] &
\end{tikzcd}$$

<u>Exml</u>: a full exceptional sequence $E_1 — E_n$ gives a semi-orth. decomp. $\langle E_1 \rangle \ldots \langle E_n \rangle$ of $\mathcal{D}$.

__Def.__ We say that exceptional collection is __strong__ if we don't allow any higher homs at all, i.e.

$$\mathrm{Hom}^i(E_k, E_l) = 0 \quad \forall\, k,l;\ i>0.$$

__Expl:__ Beilinson collection on $D^b(\mathbb{P}^n)$.

$$\langle \mathcal{O}(a), \mathcal{O}(a+1), \ldots \mathcal{O}(a+n)\rangle \rightsquigarrow \text{full strong exceptional collection}$$

always the toughest part is to show fullness, i.e. that $\langle E_1, - E_n\rangle^{\perp} = 0$ in $\mathcal{D}$, for this he introduces special spectral sequence that works in a case of $\mathbb{P}^n$.

Semi-orthogonal decomposition for blow-ups.

Let $\varepsilon : \tilde{X} \to X$ be a blow-up along $Y \subset X$ smooth, codim c

we have natural maps $\iota : E \hookrightarrow \tilde{X}$; $\pi : E = \mathbb{P}(\mathcal{N}) \to Y$, where E is except. locus.

$$\Phi_k := \iota_* \circ \left(\mathcal{O}_E(kE) \otimes \pi^*(-)\right) : D^b(Y) \to D^b(\tilde{X})$$

everything is derived

$$\begin{tikzcd} & D^b(E) \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & \\ D^b(Y) \arrow[rr] & & D^b(\tilde{X}) \end{tikzcd}$$

Let $\quad \mathcal{D}_k = \mathrm{Im}(\Phi_{-k})$; $\mathcal{D}_0 = \varepsilon^* D^b(X)$

Thm (Orlov) $\quad D^b(\tilde{X}) = \langle \mathcal{D}_{-c+1}, -, \mathcal{D}_0 \rangle$

That motivated him for the following question:

Question (Orlov) Let X be sm. pr. var$/k$. If $D^b(X)$ admits a full exc. collection $\Rightarrow$ X is rational.

Exmpl: Holds for sm. pr. toric varieties over general field.

Fact: There are no full exceptional collections on CY varieties.

§ King's conjecture:

Conj.: X sm. pr. toric variety $\Rightarrow$ X has a tilting sheaf which is a sum of line bndls.

Def. A tilting sheaf $\mathcal{T}$ is $\mathcal{O}_X$-module s.t.:

(i) $\mathrm{Hom}^i(\mathcal{T}, \mathcal{T}) = 0 \quad i \geq 1$

(ii) $\mathrm{Hom}(\mathcal{T}, \mathcal{T})$ has finite global dim

(iii) $\mathcal{T}$ generates the derived category

line bundles

In our case we also require that $\mathcal{T} = \bigoplus \mathcal{L}_i$

$\Rightarrow$ $\mathcal{L}_1$ — $\mathcal{L}_n$ makes a strong exc. collection.

Indeed, we have that

$$\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{L}_j \otimes \mathcal{L}_i^\vee) = H^0(X, \mathcal{L}_j \otimes \mathcal{L}_i^\vee),$$

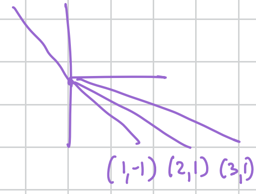
Since one of $H^0(X, \mathcal{L}_j \otimes \mathcal{L}_i^\vee)$ or $H^0(X, \mathcal{L}_j^\vee \otimes \mathcal{L}_i)$ must be vanishing we can order $\mathcal{L}_i$ properly.

Counterexample:

denote X

Take Hirzebruch surface $\mathbb{F}_2$ and blow it up in 3 pts.

If there is a strong exc. collection it always descents to the generators of $K_0(X)$.



In our case we have that $K_0(\mathbb{F}_2)$ has 4 generator + 3 generators from blow-ups.

So we are looking for a strong exc. collection of length 7.

Note that we can always twist exc. collection by line bundle, so we assume it has $\mathcal{O}_X$. It means that

$$H^i(X, \mathcal{L}_j) = 0 \quad \& \quad H^i(X, \mathcal{L}_j^\vee) = 0 \qquad i > 0.$$

pretty strong condition

So essentially how they proceed with the classification of all line bundles on X satisf. this property (infinitely many) and show that they are not compatible.

<u>Fact</u>: $\mathbb{F}_2$ has a strong exc. collection

$$\langle \mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1) \rangle.$$

<u>Thm</u> (Kawamata). X sm. pr. toric v. $\Rightarrow \exists$ full exc. collection of sheaves

drops "strong", "of line bundles".

Disclaimer: in the paper we read we consider derived categories on stacks.