Derived categories of cubic fourfolds

Bogdan Simeonov

Overview

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- Cubic fourfolds, their moduli space and the rationality problem

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- Hodge theoretic vs derived categorical perspective

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- Hodge theoretic vs derived categorical perspective
- Example: the class of cubics containing a plane

Rationality of cubic fourfolds

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What about cubic fourfolds? There is some locus of cubic fourfolds which are rational, but it is (conjecturally) neither open nor closed! Let's begin to understand their moduli space via the Torelli theorem of Voisin:

Theorem (Voisin): *The period map*

$$\mathcal{C} \to \mathcal{D}$$

sending a cubic fourfold X *to* $\{H^{1,3}(X) \subset H^4(X, \mathbb{C})\}$ *is an open immersion.*

Hodge theory cubic fourfolds

Hodge theory of the cubic fourfold

Let $X \subset \mathbb{P}^5$ be the zero locus of a degree 3 polynomial.

Hodge theory of the cubic fourfold

Let $X \subset \mathbb{P}^5$ be the zero locus of a degree 3 polynomial. We can compare its Hodge diamond to that of a K3 surface:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Cubic fourfold

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

K3 surface

If we take the primitive cohomology of the cubic fourfold, we get $(0, 1, 20, 1, 0)$ which looks exactly like the middle cohomology of a K3 surface!

Cubic fourfolds and K3 surfaces

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
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For *special cubic fourfolds* we can find a class $T \in H^{2,2}(X, \mathbb{Z})$ and move to its orthogonal complement (a generic cubic has no such class!)

Hasset's theorem

Because of the Torelli theorem, finding the class of cubic fourfolds for which such a surface exists and moreover has an associated K3 surface is a lattice-theoretic problem:

Theorem (Hasset): *The cubics containing an integral class T as above form a family of irreducible divisors $\mathcal{C}_d$, nonempty iff (*) $d > 6, d = 0, 2 \pmod 6$. Moreover, the cubics in $\mathcal{C}_d$ have an associated K3 surface i.e. $\exists S$ such that*

$$H^2_{prim}(S; \mathbb{Z})(-1) \simeq \langle h^2, T \rangle^{\perp}$$

precisely when (**) d is not divisible by four, nine, or any odd prime $p = -1 (\bmod 3)$

Derived perspective

The derived perspective

Kuznetsov presented a different viewpoint on the cubic fourfold using derived categories. The three line bundles $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$ form an exceptional collection and he defined the component

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^\perp \subset \mathcal{D}(X)$$

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Kuznetsov showed that $\mathcal{A}_X$ is a *Calabi-Yau 2* category and posed the conjecture:

> **Conjecture (Kuznetsov):** *X is rational if and only if there is a K3 surface S such that*
>
> $$\mathcal{D}(S) \simeq \mathcal{A}_X$$

We will embark to prove that the Kuznetsov component is CY2 and explore this conjecture in the case of cubics containing a plane. But before that, perhaps we need a refresher?

Kuznetsov component is CY2

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We consider the general case of a hypersurface in projective space $X \subset \mathbb{P}^{n+1}$.

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$$\mathcal{S}_{\mathcal{A}_X}^{-1} = \mathbf{L}_{\langle \mathcal{O},\dots,\mathcal{O}(n+1-d)\rangle} \circ \mathcal{S}_X^{-1} = \mathbf{L}_{\langle \mathcal{O},\dots,\mathcal{O}(n+1-d)\rangle} \circ (- \otimes \omega^{-1})[-n]$$

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But

$$\mathbf{L}_{\langle \mathcal{O}, \dots, \mathcal{O}(n+1-d) \rangle} \simeq \mathbf{L}_{\mathcal{O}} \circ \cdots \circ \mathbf{L}_{\mathcal{O}(n+1-d)}$$

If we put

$$\mathbf{O} := \mathbf{L}_{\mathcal{O}} \circ (- \otimes \mathcal{O}(1))$$

then we can see that

$$\mathbf{O}^{n+2-d} \simeq \mathbf{L}_{\mathcal{O}} \circ \mathbf{L}_{\mathcal{O}(1)} \circ \dots \circ \mathbf{L}_{\mathcal{O}(n+1-d)} \circ (- \otimes \mathcal{O}(n+2-d)) =$$
$$= \mathbf{L}_{\langle \mathcal{O}, \dots, \mathcal{O}(n+1-d) \rangle} \circ (- \otimes \omega^{-1}) \simeq \mathcal{S}_{\mathcal{A}_X}^{-1}[n]$$

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Similarly, after tensoring by $\mathcal{O}(1)$ we get

$$T = \Phi_{[\mathcal{O}(1) \boxtimes \mathcal{O} \to \mathcal{O}_\Delta(1)]}$$

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Proposition : *The kernel K^i fits into a sequence*

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$$(\mathcal{O}(1) \boxtimes \mathcal{O}) \circ K \to (\mathcal{O}(1) \boxtimes \mathcal{O}) \circ (\mathcal{O}(1) \boxtimes \mathcal{O}) \to (\mathcal{O}(1) \boxtimes \mathcal{O}) \circ \mathcal{O}_\Delta(1)$$
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The first is $\mathcal{O}(1)\boxtimes$ an Euler sequence, so we see that

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We can now replace this in the third triangle, as well as $\mathcal{O}(1) \circ K$ by the 2-term complex to get

$$K \circ K \to \mathcal{O}(1) \boxtimes \Omega(1) \to \mathcal{O}(2) \boxtimes \mathcal{O} \to \mathcal{O}_\Delta(2)$$

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Now let us write the complex giving the kernel K^d:

$$\mathcal{O}(1) \boxtimes \Omega^{d-1}(d-1) \to \cdots \to \mathcal{O}(d) \boxtimes \mathcal{O} \to \mathcal{O}_{\Delta_X}(d)$$

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If we let K' be everything except the last $\mathcal{O}_{\Delta_X}(d)$ then we have an exact triangle:

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There is also an exact triangle, coming from Koszul resolving $X \times X$ in $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$ and then passing to cohomology sheaves:

$$\mathcal{O}_{\Delta_{\mathbb{P}}}(d)|_{X \times X} \longrightarrow \mathcal{O}_{\Delta_X}(d) \longrightarrow \mathcal{O}_{\Delta_X}[2]$$

by using the Beilinson resolution of the diagonal tensored with $\mathcal{O}(d) \boxtimes \mathcal{O}$ we will show that $\Phi_{K'} \simeq \Phi_{\mathcal{O}_{\Delta_{\mathbb{P}}}(d)|_{X \times X}}$

Kuznetsov component is Calabi-Yau

The pullback of Beilinson's resolution to $X \times X$ is going to have more terms than the kernel K' however they all act trivially on $\mathcal{A}_X$:

$$\underbrace{\mathcal{O}(d-n-1) \boxtimes \Omega^{n+1}(n+1) \to \cdots \to \mathcal{O} \boxtimes \Omega^d(d)}_{\text{extra stuff in Beilinson resolution}}$$

$$\underbrace{\to \mathcal{O}(1) \boxtimes \Omega^{d-1}(d-1) \to \cdots \to \mathcal{O}(d) \boxtimes \mathcal{O}}_{K'}$$

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For $\mathcal{E}_i = \mathcal{O}(d-i) \boxtimes \Omega^i(i)$ with $i = d, d+1, \ldots, n+1$, its Fourier-Mukai transform on $A \in \mathcal{A}_X$ vanishes, by using the projection formula, base change and the fact that $\mathcal{A}_X$ is orthogonal to $\mathcal{O}, \ldots, \mathcal{O}(n+1-d)$:

$$\Phi_{\mathcal{E}_i}(A) = q_*(p^*A \otimes p^*\mathcal{O}(d-i) \otimes q^*\Omega^i(i)) =$$
$$\Omega^i(i) \otimes q_*p^*A(d-i) = \Omega^i(i) \otimes \mathbf{Hom}(\mathcal{O}(i-d), A) = 0$$

We conclude that $\Phi_{K'} \simeq \Phi_{(\iota\times\iota)^*\mathcal{O}_{\Delta_{\mathbb{P}}}(d)}$ and hence $T^d = [2]$.

Kuznetsov component is Calabi-Yau

With all of this in mind, we can finally show:

Proposition (Kuznetsov component of cubic fourfold): *We have that*

$$\mathcal{S}_{\mathcal{A}_X} = [2]$$

for a cubic fourfold. Hence, it is a Calabi-Yau 2-category.

One should think of this as a non-commutative K3 surface.

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Cubics containing a plane and twisted K3 surfaces

Fundamental (non)-example: cubics containing a plane

The first nonempty item on Hasset's list consists of the cubics containing a plane. This is $\mathcal{C}_8$ (8 is coming from the intersection matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$).

Fundamental (non)-example: cubics containing a plane

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This class of cubics is perhaps the most important one: Voisin used it to prove the Torelli theorem for cubic fourfolds, and Addington-Thomas showed that $\mathcal{C}_8 \cap \mathcal{C}_d \neq \emptyset$ for all other nonempty $\mathcal{C}_d$ and used deformation theory out of $\mathcal{C}_8$ to deduce that Hasset's and Kuznetsov's conjectures are in fact equivalent!

Cubics containing a plane

If $P \subset X \subset \mathbb{P}^5$ is a cubic fourfold containg a plane, then there is a residual quadric fibration More on the geometry

$$\mathrm{Bl}_P X = \tilde{X} \to \mathbb{P}^2$$

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If we define S to be the space of rulings on the fibers, we see that it is a double cover of $\mathbb{P}^2$ branched along the sextic curve. Moreover, the relative Fano variety parametrizing lines in the fibers is a $\mathbb{P}^1$ bundle over this:

$$\widetilde{\mathscr{F}} \xrightarrow{\pi_S} S \xrightarrow{\pi} \mathbb{P}^2$$

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We can consider *twisted vector bundles* on S which only satisfy the cocycle condition up to α, and they have a derived category $\mathcal{D}(S, \alpha)$.

Theorem (Bernardara's twisted projective bundle formula): *There is a semiorthogonal decomposition*

$$\mathcal{D}(\widetilde{\mathscr{F}}) = \langle \mathcal{D}(S, \alpha), \mathcal{D}(S) \rangle$$

Kuznetsov's equivalence: FM kernel

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$$\mathcal{A}_X \simeq \mathcal{D}(S, \alpha)$$

One way to do this is via an FM kernel from the universal line $\widetilde{\mathscr{P}} \subset X \times \widetilde{\mathscr{F}}$ with ideal sheaf $\mathcal{I}$ and its adjoint:

$$\begin{tikzcd}
\mathcal{D}(X) \arrow[r, bend left, "\Phi"] & \mathcal{D}(\widetilde{\mathscr{F}}) \arrow[l, bend left, "\Psi"] \\
\mathcal{A}_X \arrow[u, hook] & \mathcal{D}(S, \alpha) \arrow[l, "\simeq"] \arrow[u, hook]
\end{tikzcd}$$

Kuznetsov's equivalence: mutation

The idea is to compare two different SOD's for $\tilde{X}$: one coming from its realization as a quadric fibration, the other as a blowup. We turn the quadric fibration one to the blowup one in steps, which transports a twisted category $\mathcal{D}(\mathbb{P}^2, \mathcal{B})$ into the Kuznetsov component.

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$$\langle \Phi\mathcal{D}(\mathbb{P}^2, \mathcal{B}), \quad \mathcal{O}(-h), \quad \mathcal{O}, \quad \mathcal{O}(h), \quad \mathcal{O}(H), \quad \mathcal{O}(h+H), \quad \mathcal{O}(2h+H)\rangle$$

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Now, we move the line bundle $\mathcal{O}(-h)$ all the way around:

$$\langle \mathcal{O}(-h), \quad \Phi'\mathcal{D}(\mathbb{P}^2, \mathcal{B}), \quad \mathcal{O}, \quad \mathcal{O}(h), \quad \mathcal{O}(H), \quad \mathcal{O}(h+H), \quad \mathcal{O}(2h+H)\rangle$$

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Kuznetsov's equivalence: mutation

We transpose the line bundles at the end, which we can do as their Ext groups vanish and they are completely orthogonal:

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This is done via a projection formula:

$$\begin{gathered}
\mathrm{Ext}^\bullet_{\tilde{X}}(\mathcal{O}(2h+H), \mathcal{O}(2H)) \simeq \mathrm{Ext}^\bullet_{\tilde{X}}(\mathcal{O}, \mathcal{O}(H-2h)) \simeq \\
\mathrm{Ext}^\bullet_{\mathbb{P}^2}(\mathcal{O}, \phi_* \mathcal{O}(H-2h)) \simeq \mathrm{Ext}^\bullet_{\mathbb{P}^2}(\mathcal{O}, \phi_*(\mathcal{O}_\phi(1) \otimes \phi^* \mathcal{O}(-2)) \simeq \\
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Kuznetsov's equivalence: mutation

Left mutate the twisted derived category:

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Simultaneously right mutate a bunch of line bundles:

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To compute the mutations, we need to use the exact triangle

$$\mathbf{R}_{\mathcal{O}}(\mathcal{O}(h-H)) \to \mathcal{O}(h-H) \to \mathbf{Hom}(\mathcal{O}(h-H), \mathcal{O})^* \otimes \mathcal{O}$$

together with the fact that $E \sim H - h$ and the fact that $\mathrm{Ext}^\bullet(\mathcal{O}(h-H), \mathcal{O}) \simeq \mathbb{C}$. This follows again by a projection formula argument.

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The result is:

$$\langle \Phi'' \mathcal{D}(\mathbb{P}^2, \mathcal{B}), \quad \mathcal{O}, \quad \mathcal{O}_E, \quad \mathcal{O}(H), \quad \mathcal{O}_E(H), \quad \mathcal{O}(2H), \quad \mathcal{O}_E(2H) \rangle$$

Finally, we do a transposition (the two line bundles are completely orthogonal) and a right mutation to put it in the following form:

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But this looks the same as the semiorthogonal decomposition given by the blowup formula:

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We finally deduce:

Theorem (Kuznetsov): *For cubics X containing a plane, we have the equivalence*

$$\mathcal{A}_X \simeq \mathcal{D}(\mathbb{P}^2, \mathcal{B}) \simeq \mathcal{D}(S, \alpha)$$

Relation to other cubic fourfolds

We have just seen that the cubic fourfolds with a plane behave like twisted K3 surfaces. A natural question to ask is when are they actually geometric, i.e. $\alpha = 1$?

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Theorem (Kuznetsov): *The Brauer class vanishes precisely when there is another surface class W satisfying $\deg(W) - P \cdot W$ being odd. This is equivalent to $X \in \mathcal{C}_8 \cap \mathcal{C}_d$ where C_d is one of the divisors with associated K3 surfaces.*

Addington-Thomas' argument

Thank you for your attention!

Refresher on SOD's

Refresher on semiorthogonal decompositions, Serre functors and mutations

Recall the following fundamental notions:

Definition : *Let E be an exceptional object. Then there are two projection functors* $\mathbf{L}_E : \mathcal{D} \to \langle E \rangle^{\perp}$, $\mathbf{R}_E : \mathcal{D} \to^{\perp} \langle E \rangle$

$$\mathbf{L}_E(F) := \mathrm{cone}(\mathrm{Hom}(E, F) \otimes E \to F)$$
$$\mathbf{R}_E(F) := \mathrm{cone}(F \to \mathrm{Hom}(F, E)^{\vee} \otimes E)[-1]$$

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This definition can be extended to so-called admissible subcategories, where there are more general replacements for the functors $\mathrm{Hom}(E, -), \mathrm{Hom}(-, E)^{\vee} : \mathcal{D} \to \langle E \rangle$ adjoint to the inclusion.

Refresher on semiorthogonal decompositions, Serre functors and mutations

Proposition (Mutations of exceptional sequences): *A full exceptional sequence $\langle E_1, \ldots, E_n \rangle$ gives rise to mutated full exceptional sequences*

$$\langle E_1, \ldots, E_{i-1}, \mathbf{L}_{E_i} E_{i+1}, E_i, E_{i+2}, \ldots, E_n \rangle$$

and

$$\langle E_1, \ldots, E_{i-2}, E_i, \mathbf{R}_{E_i} E_{i-1}, E_{i+1}, \ldots, E_n \rangle.$$

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Definition (Serre functor): *A Serre functor is an additive, $\mathbb{C}$-linear autoequivalence obeying:*

$$\mathrm{Hom}(A, B)^* \simeq \mathrm{Hom}(B, S(A))$$

A category is CYn if the Serre functor is given by $[n]$. back

Cubics containing a plane: geometry

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Projecting orthogonally out of a plane $P \subset \mathbb{P}^5$ gives a rational map

$$\begin{tikzcd}
\mathbb{P}\mathcal{N}_{P/\mathbb{P}^5} \simeq E \arrow[r] \arrow[d] & \mathrm{Bl}_P\mathbb{P}^5 \arrow[d, "\tau"] \arrow[dr, "\phi"] & \\
P \arrow[r] & \mathbb{P}^5 \arrow[r, dotted] & \mathbb{P}^2
\end{tikzcd}$$

In coordinates, if $P = \mathbf{V}(z_0, z_1, z_2)$, then the rational map is given by the linear system $|\mathcal{I}_P \otimes \mathcal{O}(1)|$ which has basis z_0, z_1, z_2 and base locus P.

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Hence, we see that

$$\mathcal{I}_E \otimes \tau^*\mathcal{O}(1) \simeq \phi^*\mathcal{O}(1)$$

If we denote by H the hyperplane class in $\mathbb{P}^5$ and by h the one on $\mathbb{P}^2$, then this implies that

$$E \sim H - h$$

Cubics containing a plane: geometry

If we write the blowup in coordinates, we see that

We can think of the blowup in this simplified setting as the set

$$\mathrm{Bl}_P\mathbb{P}^5 = \{p \in \mathbb{P}^5, q \in \mathbb{P}^2 |\, p_i q_j = p_j q_i, 0 \leqslant i, j \leqslant 2\} \subset \mathbb{P}^5 \times \mathbb{P}^2$$

which comes equipped with two projection maps to $\mathbb{P}^5$ and $\mathbb{P}^2$ which are the blowup and resolution maps, respectively. We can clearly see that over $\mathbb{P}^2$, this is a projective bundle by looking at the affine cone over $\mathbb{P}^5$:

$$\{a \in \mathbb{A}^6, q \in \mathbb{P}^2 |\, a_i q_j = a_j q_i, 0 \leqslant i, j \leqslant 2\} \to \mathbb{P}^2$$

We can see that the coordinates a_3, a_4, a_5 are unconstrained, so give us a copy of the trivial bundle $\mathcal{O}^{\oplus 3}$. The other three coordinates give us precisely the tautological bundle:

$$\mathcal{O}(-1) = \{(a_0, a_1, a_2) \in \mathbb{A}^3, q \in \mathbb{P}^2 |\, (a_0, a_1, a_1) \in q\}$$

Hence, we conclude that

$$\mathrm{Bl}_P\mathbb{P}^5 = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 3}) := \mathbb{P}(\mathcal{F}^\vee)$$

Cubics containing a plane: geometry

The strict transform $\tilde{X} \sim 2H + h$ and is thus a quadric fibration

$$\tilde{X} \to \mathbb{P}^2$$

which geometrically can be thought of as follows: the $\mathbb{P}^2$ parametrizes 3-planes containing P, and X intersects such a 3-plane in P union a quadric.

We can see $\tilde{X}$ as the zero locus of $q \in H^0(\mathbb{P}^2, S^2\mathcal{F} \otimes \mathcal{O}(1))$ which can also be thought of as a map $\mathcal{F} \to \mathcal{F}^* \otimes \mathcal{O}(1)$. The quadric fibers are nonsingular outside the determinant locus, which is a sextic curve, since $\det(q) = \mathcal{O}(6)$. back