Notes on Geometry and Topology

Author: Bogdan Simeonov

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1 Differential Geometry

1.1 Tangent spaces

Let $p \in X$ and look at all curves $\gamma : I \to X$ with $\gamma(0) = p$. Given a chart ϕ about p, we define $\pi_p^\phi(\gamma) = (\phi \circ \gamma)'(0)$. We say that two curves agree to first order if $\pi_p^\phi(\gamma_1) = \pi_p^\phi(\gamma_2)$ for some chart ϕ. By using transition maps, we see that this is independent of the charts and hence is an equivalence relation:

$$\pi_p^\psi(\gamma) = (\psi \circ \gamma)'(0) = D(\psi \circ \phi^{-1})_{\phi(p)}(\phi \circ \gamma)'(0) = T\pi_p^\phi(\gamma)$$

T is the Jacobian matrix for the transition between the two charts. We define T_pX to be the curves at p modulo agreement to first order. π_p^ϕ gave us a surjection from curves at p to $\mathbb{R}^n$, and modding out by agreement to first order gives a bijection $T_pX \simeq \mathbb{R}^n$, allowing us to transport the vector space structure. Now, a tangent vector in one chart transforms into another chart using the Jacobians of the transition maps.

To show surjectivity, take $\gamma_v(t) = \phi^{-1}(\phi(p) + tv)$. This obviously maps to $v \in \mathbb{R}^n$. We can then define $\partial/\partial x_i = [\gamma_{e_i}]$. One should be careful, as this depends on the whole coordinate chart ϕ and not only on x_i.

Lemma 1.1 (Lemma): *We have the identity*

$$\frac{\partial}{\partial y_i} = \sum \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$$

for two different charts ϕ, ψ producing $\partial_{x_i} = (\pi_p^\phi)^{-1}(e_i)$, $\partial_{y_j} = (\psi_p^\phi)^{-1}(e_j)$.

This follows as the transition map have Jacobians, or by just evaluating both sides at the functions x_i - note that $\partial x_j/\partial y_i$ makes sense, as $x_i : X \to \mathbb{R}$ is just a function on which the tangent vector $\partial/\partial y_i$ acts on.

Let's be a bit more careful: let ϕ, ψ be charts about p in X, with respective coordinates $x_i = r_i \circ \phi$ and $y_i = r_i \circ \psi$. Notice that by definition

$$\partial x_i/\partial y_j := \partial(x_i \circ \psi^{-1})/\partial r_j = \partial(r_i \circ \phi \circ \psi^{-1})/\partial r_j$$

This is precisely the Jacobian of the transition map $\phi \circ \psi^{-1}$with respect to the standard basis. We could also have used the curve derivative definition to get

$$\frac{d}{dt}(x_i(\psi^{-1}(\psi(p) + te_j)))|_{t=0}$$

Once again, this is just $\partial x_i/\partial y_j$, but it is also the directional derivative in the direction of e_j at the point $\psi(p)$ for the transition map $\phi \circ \psi^{-1}$, i.e. $\partial(\phi \circ \psi^{-1})/\partial r_j$.

Given $[\gamma]$ a tangent vector and a smooth $f : X \to \mathbb{R}$, then $\gamma \cdot f = (f \circ \gamma)'(0)$ is a real number. Note that this does not depend on the choice of γ in the equivalence class by using the chain rule.

1.1.1 Derivatives

The differential of a smooth map $F : X \to Y$ is a map between tangent spaces $D_pF : T_pX \to T_{F(p)}Y$ defined by $D_pF([\gamma]) = [F \circ \gamma]$. To calculate this in coordinates, one can just observe that $DF_p(\partial/\partial x_i) = \sum a_{ij}\partial/\partial y_j$and evaluate both sides at the function y_i. Hence $a_{ij} = \partial(y_j \circ F)/\partial x_i$. Another way to see this is the following:

$$D_pF(\partial_{x_i}) = D_pF((\pi_p^{\phi})^{-1}(e_i)) = (\pi_{F(p)}^{\psi})^{-1}(Te_i)$$

This is since $\pi^{\psi}(F \circ \gamma) = (\psi \circ F \circ \gamma)'(0) = (\psi \circ F \circ \phi^{-1} \circ \phi \circ \gamma)'(0) = T\pi^{\phi}(\gamma)$, where T is the matrix representing $\tilde{F}$. But this matrix is just the Jacobian

$$(\partial \tilde{F}_j/\partial r_i) = (\partial(\psi \circ F)_j/\partial x_i) = (\partial(y_j \circ F)/\partial x_i)$$

Hence

$$D_pF(\partial_{x_i}) = (\pi_{F(p)}^{\psi})^{-1}(\sum \frac{\partial(y_j \circ F)}{\partial x_i} e_j) = \sum \frac{\partial(y_j \circ F)}{\partial x_i} \frac{\partial}{\partial y_j}.$$

This actually allows us to write $[\gamma] = d\gamma_0(\partial/\partial t) \in T_pX$. We get the chain rule almost by definition.

Remark: we have a map

$$\mathrm{Diff}(M) \to \mathrm{Aut}(C^{\infty}(M))$$
$$\phi \mapsto \phi^*$$

The differential of this is a map from tangent vectors of $\mathrm{Diff}(M)$ to tangent vectors of $\mathrm{Aut}(C^{\infty}(M))$. But note that a tangent vector in $\mathrm{Diff}(M)$ is given by the derivative of a curve of diffeomorphisms, which is precisely the flow of a vector field on M. Hence, it is a vector field, i.e. an element of $\mathfrak{X}(M)$. On the other hand, a tangent vector on $\mathrm{Aut}(C^{\infty}(M))$ is precisely a derivation on M. All in all, the resulting differential is

$$\mathfrak{X}(M) \to \mathrm{Der}(M)$$
$$X \mapsto \mathcal{L}_X$$

Proposition 1.2 (Derivations and vector fields): *The map $X \mapsto \mathcal{L}_X$ is an isomorphism between $\Gamma(TM)$ and* $\mathrm{Der}(M)$.

1.2 Immersions, submersions, local diffeomorphisms

Depending on the linear map DF_p, we either get an injection, surjection, or both - these correspond to being an immersion, submersion or local diffeomorphism. The last part follows by the inverse function theorem. Note that all these conditions are open. Note that one can choose local coordinates so that the derivative of a local diffeomorphism looks like the identity matrix.

The submersion and immersion theorems say that there exist local coordinates such that F looks like a projection, respectively inclusion of $\mathbb{R}^n$.

1.3 Submanifolds

We defined them as vanishing loci of a bunch of coordinates ,i.e. $\{z_1 = \ldots = z_k = 0\}$ is a codimension k submanifold. We then get local coordinates $(z_{k+1}, \ldots, z_n)$ on this, so it is indeed a manifold.

Using the submersion theorem, one can show that given a regular value q where a map $F : X \to Y$ is a submersion, then the subset $F^{-1}(q)$ is a submanifold of X of codimension equal to the dimension of Y.

Theorem 1.3 (Sard's theorem): *Regular values are dense, critical values form a measure zero subset (E.g. $\mathbb{R} \subset \mathbb{R}^{\nvDash}$).*

1.4 Transversality

Definition 1.4 (Transversality): *$Y, Z \subset X$ are transverse at p if $T_pX + T_pZ = T_pX$. Equivalently, $ann(T_pY) \cap ann(T_pZ) = 0$ in the dual space, i.e. the equations cutting up Y and Z are independent of each other.*

Proposition 1.5 (Dimension of transverse intersection): > *Y and Z intersect transversely $\implies$ $Y \cap Z$ is a submanifold of X of codimension the sum of codimensions of Y and Z.*

Proof. We have local coordinates y and z (maps from X to $\mathbb{R}^n$) which define Y and Z as the zero loci of the first k resp. first l components. Now define $f : U \to \mathbb{R}^{k+l}$, $a \mapsto (y_1 a, \ldots, y_k a, z_1 a, \ldots, z_l a)$. Then the kernel of df_p consists of $T_pX \cap T_pY$, so df_p must be surjective by the transversality assumption. In other words, we can think of this as

$$T_pU = T_pX \xrightarrow{df_p} T_pX/T_pZ \oplus T_pX/T_pY$$

This is because: $\langle \partial/\partial z_1, \ldots, \partial/\partial z_n \rangle = T_pX$, $\langle \partial/\partial z_{l+1}, \ldots, \partial/\partial z_n \rangle = T_pZ$, $\langle \partial/\partial y_1, \ldots, \partial/\partial z_n \rangle = T_pX$, $\langle \partial/\partial y_{k+1}, \ldots, \partial/\partial z_n \rangle = T_pY$. In particular, the map is surjective, so we have a submersion at all such points p in the intersection, i.e. we get a coordinate system where the vanishing of the $k+l$ coordinates gives us the intersection of Y and Z: $(x_1, \ldots, x_k) = (y_1, \ldots, y_k); (x_{k+1}, \ldots, x_{k+l}) = (z_1, \ldots, z_l)$. Basically, this is built up from two maps $U \to \mathbb{R}^k, U \to \mathbb{R}^l$ both of whose effects on the tangent spaces kill of the tangent space at Z resp. Y. Informally: $df(\partial/\partial z_i) = \partial f/\partial z_i =$

$\partial(y_1,...,y_k,z_1,...,z_l)/\partial z_i = \partial/\partial z_i$ provided $i \leq l$, and similarly for $df(\partial/\partial y_i)$, since in a way these should be independent and the chain rule can be ignored. $\square$

1.5 Vector bundles

Usual definition using trivializing cover and maps Φ_α. If $(U_\alpha \cap U_\beta) \times \mathbb{R}^k \xrightarrow{\Phi_\beta \circ \Phi_\alpha^{-1}} (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is given by $(b, v) \mapsto (b, g_{\beta\alpha}(b)(v))$. Then $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ satisfies the cocycle condition

1.5.1 The tangent bundle

$$TX = \coprod T_p X$$

The trivialization of this is given by: $TU = \coprod_{p \in U} T_p X \to U' \times \mathbb{R}^n$ given by $(p, \sum a_i \partial_{x_i}) \mapsto (p', a_1, ..., a_n)$. Hence, the tangent bundle is locally trivial on the charts.

Note that a section of this, i.e. a vector field, is smooth if and only if the functions a_i are smooth.

Representing global sections:

$Hom(\underline{\mathbb{R}}, E) \simeq \Gamma(E)$, given by $G \mapsto G(-, 1)$ and $s \mapsto G(b, t) = ts(b)$

Cocycles give vector bundles:

$E = \coprod U_\alpha \times \mathbb{R}^k / \sim$, where $(b, v) \sim (b, g_{\beta\alpha}(b)v)$, $b \in U_\alpha \cap U_\beta$.

1.5.2 Tangent bundles of quotient manifolds

Given a manifold X and an action $G \times X \to X$ then the tangent space of X/G at $\pi(x)$ is given by

$$T_{\pi(x)} X/G = T_x X / T_x(G \cdot x)$$

This is since the orbit $G \cdot x$ is an embedded submanifold of the dimension of G inside X and its tangent space gets killed of by the projection $d\pi_x$: if $\gamma(t) = g_t \cdot x$ is a representing curve for a tangent vector in $G \cdot x$,then

$$d\pi_x(\dot{\gamma}) = \frac{d}{dt}|_{t=0} \pi(\gamma(t)) = 0$$

Since this has the right dimension, we get that the submersion

$$d\pi_x : T_x X \to T_{\pi(x)} X/G$$

has kernel exactly $T_x(G \cdot x)$. Note that this is spanned by the fundamental vector fields

$$\xi_A(x) := \frac{d}{dt}|_{t=0} \exp(tA) \cdot x$$

for $A \in \mathfrak{g}$.

Example (Example): Complex tori have trivial tangent bundles.

Example (Tangent bundle of projective space): For the projective space $\mathbb{CP}^n = S^{2n+1}/U(1)$ we have that if l is spanned by a unit vector v then

$$T_l\mathbb{CP}^n = T_vS^{2n+1}/T_v(U(1)\cdot v)$$

But $T_vS^{2n+1} = \{(v,w)|\langle v,w\rangle = 0\} = l^\perp$ (note that this is the real orthocomplement). Moreover, since $U(1) = S^1$ its tangent bundle is spanned by ∂_θ which is the section $\partial_\theta : S^1 \to TS^1$ which at $e^{i\alpha}$ gives $(e^{i\alpha}, ie^{i\alpha})$, since the tangent space is given by orthogonal vectors. Its exponential is given by the time flow of the integral curve $\gamma(t) = e^{it}$ i.e. $\dot{(\gamma(t))} = ie^{it} = \partial_\theta(\gamma(t))$. Hence, $T_v(U(1)\cdot v)$ is spanned by the fundamental vector

$$\frac{d}{dt}|_{t=0}\exp(t\partial_\theta)\cdot v = \frac{d}{dt}|_{t=0}e^{it}\cdot v = iv$$

so the tangent space is given by the imaginary multiples of v. One can show that this is precisely $\mathrm{Hom}(l, l^\perp)$.
Another way to do this is to notice that the action

$$\mathrm{GL}(n+1,\mathbb{C})\times\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$$

descends to $\mathbb{CP}^n$. Thus, fixing a line l we get a map

$$\mathrm{GL}(n+1,\mathbb{C})\to\mathbb{CP}^n$$

which is a submersion. On tangent spaces, we get

$$Mat(n+1\times n+1,\mathbb{C}) = T_I\mathrm{GL}(n+1,\mathbb{C})\to T_l\mathbb{CP}^n$$

The kernel of this map consists of the matrices M which preserve l: $Ml\subset l$. Hence, $T_l\mathbb{CP}^n$ is isomorphic to all matrices modulo this kernel, which can be identified with $\mathrm{Hom}(l, l^\perp)$.
More generally, the same trick can be done for Grassmanians: the action

$$\mathrm{GL}(V)\times V\to V$$

descends to

$$\mathrm{GL}(V)\times\mathrm{Gr}_k(V)\to\mathrm{Gr}_k(V)$$

Hence, fixing $W\in\mathrm{Gr}_k(V)$, we get a submersion $\mathrm{GL}(V)\to\mathrm{Gr}_k(V)$ and hence the tangent space at W can be identified with the endomorphisms of V modulo the ones that preserve W:

$$T_W\mathrm{Gr}_k(V) = \frac{\mathrm{End}(V)}{\{M|MW\subset W\}}\simeq\mathrm{Hom}(W, V/W)$$

To see the last isomorphism, consider the following composition:

$$\mathrm{End}(V)\xrightarrow{restriction}\mathrm{Hom}(W,V)\xrightarrow{quotient}\mathrm{Hom}(W,V/W)$$

This is surjective and has kernel precisely the endomorphisms which when restricted to W end up in W, i.e. the endomorphisms which preserve W.

1.5.3 Operations on VB's

- Duals: cocycles given by $(g_{\beta\alpha}^T)^{-1}$
- Tensor products: Kronecker product of cocycle matrices
- Sums: sums of cocycle matrices
- Exterior powers

1.5.4 Cotangent bundle

Dually to how we defined the tangent vectors as equivalence classes of jets, we can define the cotangent bundle as equivalence classes of maps $f : X \to \mathbb{R}$. The equivalence is agreement to first order, i.e. $df_p = dg_p$. The idea is that if they agree to first order, they will act the same on tangent vectors. We will show this is the same as $T_p^* X$.

We have a map from functions to $T_p^* X$ given by $[f] \mapsto df = \sum \partial f / \partial x_i \, dx_i$. This is surjective, as dx^j is a dual basis to the basis of tangent vectors ∂_{x_j}.

Example (Transition maps for the cotangent bundle):

$$dx^i = \sum \frac{\partial x^i}{\partial y^j} dy^j$$

Hence, the transition matrix $\frac{\partial x^i}{\partial y^j}$ is the inverse transpose of the one of the tangent bundle, which is $\frac{\partial y^j}{\partial x^i}$.

Lemma 1.6 (Pullbacks commute with differentials): *Pullbacks commute with differentials*

Proof. Given $F : X \to Y$, we show that its pullback commutes with d. The pullback is given by the dual of the differential dF. Now let $g : Y \to \mathbb{R}$. Then:

$$F^* dg([\gamma]) = dg(dF[\gamma]) = dg([F \circ \gamma]) = (g \circ F \circ \gamma)'(0) = d(g \circ F)[\gamma] = d(F^* g)[\gamma]$$

□

1.6 Differential forms

Let $\alpha = \alpha_I dx^I$ be a differential form in local coordinates. We would like to somehow get a 2-form from α. Naively, we could try:

$$d\alpha = \frac{\partial \alpha_s}{\partial x^l} dx^l \otimes dx^s$$

However, this does not transform correctly using a different trivialization: if we use coordinates y^i, then $\alpha = \alpha'_i dy^i$ and we know that $dy^i = \sum \frac{\partial y^i}{\partial x^l} dx^l$. Hence $\alpha_s = \alpha'_i \frac{\partial y^i}{\partial x^s}$ and

$$d\alpha = \frac{\partial \alpha_s}{\partial x^l} dx^l \otimes dx^s = \frac{\partial}{\partial x^l}(\alpha'_i \frac{\partial y^i}{\partial x^s}) dx^l \otimes dx^s = [\frac{\partial \alpha'_i}{\partial x^l}\frac{\partial y^i}{\partial x^s} + \alpha'_i \frac{\partial^2 y^i}{\partial x^l \partial x^s}] dx^l \otimes dx^s$$

But by the chain rule:

$$\frac{\partial \alpha'_i}{\partial y^j} dy^j \otimes dy^i = \frac{\partial \alpha'_i}{\partial y^j}\frac{\partial y^j}{\partial x^l} dx^l \otimes \frac{\partial y^i}{\partial x^s} dx^s = \frac{\partial \alpha'_i}{\partial x^l}\frac{\partial y^i}{\partial x^s} dx^l \otimes dx^s$$

Hence, what we end up with is:

$$\frac{\partial \alpha'_i}{\partial x^l}\frac{\partial y^i}{\partial x^s} dx^l \otimes dx^s + \alpha'_i \frac{\partial^2 y^i}{\partial x^l \partial x^s} dx^l \otimes dx^s = \frac{\partial \alpha'_i}{\partial y^j} dy^j \otimes dy^i + junk$$

Note that the first bit is precisely the desired $d\alpha$ in y-coordinates, so we need to somehow get rid of the junk - enter alternating algebra! If we replaced all the tensors with $\wedge$, then the junk will cancel out.

Definition 1.7 (Exterior derivative): *If $\alpha = \sum \alpha_I dx^I$, then $d\alpha = \sum \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I = d\alpha_I \wedge dx^I$. This has the following properties:*

- *It is $\mathbb{R}$-linear, agrees with d on 0-forms*
- $d^2 = 0$
- $F^* d = dF^*$
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$

Proof.

$$d(\alpha \wedge \beta) = d(\alpha_I \beta_J dx^I dx^J) = d(\alpha_I \beta_J) dx^I dx^J =$$
$$= (d\alpha_I \beta_J + \alpha_I d\beta_J) dx^I dx^J = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$$

$$F^* d\alpha = F^*(d\alpha_I \wedge dy^{i_1} \wedge \ldots \wedge dy^{i_p}) = F^* d\alpha_I \wedge F^* dy^{i_1} \wedge \ldots \wedge F^* dy^{i_p} =$$
$$= dF^* \alpha_I \wedge dF^* y^{i_1} \wedge \ldots \wedge dF^* y^{i_p} = d(F^* \alpha_I \wedge dF^* y^{i_1} \wedge \ldots \wedge dF^* y^{i_p}) = dF^* \alpha$$

□

1.7 Integration and orientation

An orientation on a vector space V is a choice of representative modulo positive rescaling of $det(V) = \bigwedge^n V \simeq \mathbb{K}$. In the case of vector bundles, we say an orientation of $E \to X$ is a choice of nonvanishing section of the determinant line bundle $det(E)$. Hence, E is orientable if and only if $det(E)$ is trivial.

A manifold X is orientable if its tangent bundle is orientable, which holds if and only if it has a nonvanishing top differential form, which is a volume form. The volume form determines an orientation by saying $e_1, \dots, e_n$ are positively oriented if and only if $\omega(e_1, \dots, e_n) > 0$. A map is orientation preserving if its pullback preserves orientation forms, and hence its Jacobian has positive determinant.

We can now define integration of a compactly supported top form on an oriented manifold:

$$\int_X \omega = \sum \int_{\mathbb{R}^n} f_\alpha dx^1 \dots dx^n$$

We are summing over α indexing an open cover of X with an associated partition of unity and locally $\rho_\alpha \omega = f_\alpha dx_\alpha^1 \dots dx_\alpha^n$ with the x^i oriented positively(really, we are just pulling back ω to the coordinate charts - see Bott and Tu). This is well defined, since if we switch to a different coordinate system and different partition of unity, the Jacobian determinant will be positive, so changing variables does not change the sign of the integral.

Theorem 1.8 (Stokes' theorem):

$$\int_X d\omega = \int_{\partial X} \omega$$

Proof. See notes, or Bott-Tu. Uses partition of unity to reduce to Euclidean case, and then being clever with Fubini. $\square$

1.8 Connections

Given a bundle $E \to B$, we want to find a way to differentiate its sections.

Denote $\Omega^k(E) = \Gamma(E \otimes \bigwedge^k T^*B)$. In particular, $\Omega^0(E)$ consists of the sections of E over B and $\Omega^1(E)$ consists of the sections of $E \otimes T^*B \simeq \mathrm{Hom}(TB, E)$. Hence, we can think of them as 1−forms, which eat up vectors but land in E, i.e. we call them E-valued 1-forms.

Given a section $s : B \to E$, we can look at it locally in a trivializing cover $U_\alpha \subset B$. Restricting to this, we get $s|_{U_\alpha} : U_\alpha \to E|_{U_\alpha} \simeq U_\alpha \times \mathbb{R}^k$. If we compose with the right projection, we get a map $v_\alpha : U_\alpha \to \mathbb{R}^k$. Naively, we could try just differentiating this and try to glue these together to get a thing defined globally. In fact, for trivial bundles, this is completely fine. Notice that

$$dv_\alpha : TU_\alpha \to T\mathbb{R}^k \simeq \mathbb{R}^k$$

is like a vector of 1-forms, i.e. it is a vector-valued 1-form, which is kind of what we want (an E valued form, locally). Suppose the cocycle for E is given by $g_{\beta\alpha}$. Then $v_\beta = g_{\beta\alpha} v_\alpha$. We expect dv_β to transform in the same way for it to glue, i.e. we would like $dv_\beta = g_{\beta\alpha} dv_\alpha$. Notice: on the right hand side, we have a matrix of functions acting on a vector of 1-forms, which is still a vector of 1-forms, so all of this makes sense. However, this would mean that $dv_\alpha = g_{\beta\alpha}^{-1} dv_\beta$. We compute, by using Leibniz:

$$g_{\beta\alpha}^{-1} dv_\beta = g_{\beta\alpha}^{-1} d(g_{\beta\alpha} v_\alpha) = dv_\alpha + g_{\beta\alpha}^{-1} d(g_{\beta\alpha}) v_\alpha$$

To justify this, note that if g is a matrix of functions, by dg we mean the matrix with entries the 1-forms obtained by applying d to all entries. Then the Leibniz rule still follows, since if v_β^i denotes the $i-th$ component of v_β, then $v_\beta^i = g_{ij} v_\alpha^i$ and (using summation notation):

$$dv_\beta^i = d(g_{ij} v_\alpha^i) = (dg_{ij}) v_\alpha^i + g_{ij} dv_\alpha^i$$

This is a perfectly valid vector-valued 1-form: we have a matrix of 1-forms acting on a vector, giving a vector of 1-forms, and also a matrix of functions acting on a vector of 1-forms, giving a vector of 1-forms.

So our naive attempt was blocked by the obstruction term $g_{\beta\alpha}^{-1} d(g_{\beta\alpha}) v_\alpha$.To modify this and actually get a thing that transforms correctly, we introduce connections as follows:

Definition 1.9 (Connection): *A connection $\mathcal{A}$ on E comprises a $\mathfrak{gl}(k,\mathbb{R})$–valued 1-form A_α on each trivialization U_α which transforms as follows on overlaps:*

$$A_\alpha = g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1}(dg_{\beta\alpha})$$

Given this information, the covariant derivative of a section $s \in \Gamma(E)$ is defined to be

$$d^{\mathcal{A}} s = dv_\alpha + A_\alpha v_\alpha$$

This transforms correctly by direct calculation, hence it defines a map:

$$\Omega^0(E) \xrightarrow{d^{\mathcal{A}}} \Omega^1(E)$$

A section is horizontal if it is killed by the covariant derivative, i.e. $d^{\mathcal{A}} s = 0$.

Another way to think of connections is as a canonical splitting of a bundle into horizontal and vertical vectors - see the concept of Ehresmann connections.

Proposition 1.10 (Equivalent Koszul definition of connection): *The covariant derivatives coming from connections are precisely the $\mathbb{R}$-linear maps $\nabla : \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibniz rule:*

$$\nabla(fs) = s \otimes df + f\nabla s$$

This is checked by directly verifying locally (but one must also show that these are local operators - see the example sheet). Note that the Leibniz rule tells us that if $\nabla(v) = dv + Av$ locally, then A is actually a matrix-valued one-form! One can also think of the connection as $\nabla = d + A_i dx^i$ in local coordinates, where A_i is a matrix that eats a vector (a section).

Note that a trivial vector bundle has a connection given by $A_\alpha = 0$. Using this and partitions of unity, one can show that any vector bundle admits a connection.

1.8.1 Induced connections and metric connections

Given a connection ∇ on E, ∇' on E', we can define a connection on $E \otimes E'$ by the product rule

$$\nabla^{E\otimes E'}(s_1 \otimes s_2) := \nabla s_1 \otimes s_2 + s_1 \otimes \nabla' s_2$$

Moreover, we can define a connection on the dual bundle $E^\vee$ implicitly as follows:

$$d((s,t)) = (\nabla s, t) + (s, \nabla^\vee t)$$

The pairing (s,t) is the natural one between sections of E and sections of $E^\vee$ by evaluating pointwise, i.e. $(s,t) = t(s)$ is a function, of which we can take the exterior derivative and get a 1-form valued in E. This can also be seen as a sort of product rule, where the exterior derivative on the left is just the induced connection from ∇ onto the trivial bundle.

Similarly, for $\mathrm{Hom}(E, E') \simeq E^\vee \otimes E'$, one can combine the two operations above.

There is a different, equivalent way to define induced connections:

Definition 1.11 (Induced connections using representations): *Given a representation $\rho : \mathrm{GL}(k, \mathbb{R}) \to \mathrm{GL}(k, \mathbb{R})$ and a k-dimensional vector bundle $E \to B$ with cocycles g, we get a bundle $E' \to B$ with cocycles $\rho(g)$. If $\mathcal{A}$ with local connection 1-form A is a connection on E, the induced connection on E' has local connection 1-form given by $d\rho_I(A)$.*

Example (Examples): For $\rho(A) = (A^T)^{-1}$ we get the dual bundle $E^\vee$. Thus in this case,

$$d\rho_I(A) = \frac{d}{dt}_{t=0}\rho(I + tA) = \frac{d}{dt}_{t=0}((I + tA)^{-1})^T$$

But linearly, since $(I + tA)(I - tA) = I - t^2A^2$, we have that $(I + tA)^{-1} \approx I - tA$. Hence, we get $-A^T$. So if A_α defines a connection on E, then $-A_\alpha^T$ defines one on $E^\vee$, which is consistent with the implicit definition above, i.e. if the local connection matrices are given by $\nabla e_i = A_{ij}e_j, \nabla e_i^* = A_{ij}e_j^*$ then the Leibniz definition tells us

$$0 = d(e_i, e_j^*) = (\nabla e_i, e_j^*) + (e_i, \nabla e_j^*) = (A_{ik}e_k, e_j^*) + (e_i, A'_{jk}e_k^*) = A_{ij} + A'_{ji} \implies A' = -A^T$$

For $End(E)$, which has $\rho(A)(M) = AMA^{-1}$, we get that $d\rho_I(A) = \frac{d}{dt}_{t=0}\rho(I + tA) = \frac{d}{dt}_{t=0}(I + tA)(-)(I + tA)^{-1}$. Plugging in M, we get

$$\frac{d}{dt}_{t=0}(I + tA)M(I - tA) = \frac{d}{dt}_{t=0}(M - tMA + tAM - t^2AMA) = AM - MA = [A, M]$$

Hence, $d\rho_I(A) = [A, -] = L_A$, the Lie bracket.

A metric g on a bundle E is a section of the bundle $E^\vee \otimes E^\vee \simeq (E^{\otimes 2})^\vee$ which is positive definite and symmetric, i.e. an inner product on every fiber. As such, one can take its covariant derivative w.r.t. a connection on E. This implies the following implicit equality: for any sections s_1, s_2 of E,

$$(\nabla(s_1 \otimes s_2), g) + (s_1 \otimes s_2, \nabla g) = d(s_1 \otimes s_2, g)$$

Here, I am using the same notation for all induced connections (on the tensor product and the dual of the tensor product of E with itself). However, by definition $\nabla(s_1 \otimes s_2) = \nabla s_1 \otimes s_2 + s_1 \otimes \nabla s_2$, whereas the natural pairing $(s_1 \otimes s_2, g)$ is precisely evaluation, i.e. is equal to $g(s_1, s_2) = \langle s_1, s_2\rangle$(be careful not to confuse round with angled brackets). This shows that $\nabla g = 0$ precisely when

$$\langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle = d\langle s_1, s_2\rangle$$

This motivates the following:

Definition 1.12 (Compatible metrics): *A metric g is compatible with a connection ∇ if it is covariantly constant, i.e. $\nabla g = 0$ which is equivalent to the equation above.*

This has another interpretation: compatibility with the metric means that parallel transport is an isometry! The idea is as follows (see Spivak, chapter 6 for details): if s is a horizontal section along a curve c i.e. $\nabla_{\dot{c}} s = 0$, and ∇ is compatible with the metric, then $d/dt\langle s,s\rangle = 2\langle \nabla_{\dot{c}} s, s\rangle = 0$. In other words, the norm of s is constant along c. Hence, parallel transport along c is norm-preserving and hence an isometry. Conversely, if parallel transport is an isometry, then we can choose horizontal sections $P_1, ..., P_n$ which are orthonormal at a point p, but then since our assumption is that parallel transport is an isometry, they are orthonormal along the whole of c. Putting $s = s^i(t)P_i, q = q^i(t)P_i$, then $\langle s, q\rangle = s^i q^i$. Since the P_i are horizontal, we have that

$$\nabla_{\dot{c}} s = \frac{ds^i}{dt} P_i, \nabla_{\dot{c}} q = \frac{dq^i}{dt} P_i$$

All in all,

$$\langle \nabla_{\dot{c}} s, q\rangle + \langle s, \nabla_{\dot{c}} q\rangle = \frac{ds^i}{dt} q^i + s^i \frac{dq^i}{dt} = \frac{d}{dt}\langle s, q\rangle$$

Remark: the induced connection on the trivial bundle, whose sections are smooth functions of M, is precisely the usual exterior derivative, i.e. the trivial connection

Remark (Connections as an affine space over End(E)):
Connections look like things in $\Omega^1(End(E))$, but modified by a part which is given by d. This is because sections of $End(E)$ are matrix-valued functions which transform like $M_\beta = g_{\beta\alpha} M_\alpha g_{\beta\alpha}^{-1}$. In fact, the space of conections is an affine translate of $\Omega^1(End(E))$!
The idea is that a difference between two connections is going to kill the dv term, giving a thing that transforms exactly like a matrix-valued one-form.

Definition 1.13 (Curvature and the covariant exterior derivative): *A connection can be extended to a generalized covariant exterior derivative:*

$$\Omega^\bullet(E) \xrightarrow{\nabla} \Omega^{\bullet+1}(E)$$

satisfying:

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|} \omega \nabla s$$

The curvature is the square of this, i.e. $F = \nabla^2$. Hence, this forms a chain complex precisely when the curvature is zero, i.e. the connection is flat.

1.8.2 Contractions

As we will see later on, the Cartan magic formula states that $\mathcal{L}_X \omega = di_X\omega + i_X d\omega$. This is a supercommutator identity, and motivates the following definition: given a covariant derivative $\nabla : \Omega^r(E) \to \Omega^{r+1}(E)$, we can either differentiate and contract, or first contract and then differentiate:

$$\nabla_Z = i_Z \nabla + \nabla i_Z : \Omega^r(E) \to \Omega^r(E)$$

Now we can see tha, if ω is the form part and u is the E part:

$$\nabla_Z(\omega \otimes u) = i_Z(d\omega \otimes u + (-1)^r \omega \nabla u) + \nabla(i_Z \omega \otimes u) =$$
$$= \underline{(i_Z d\omega) \otimes u} + (-1)^r i_Z(\omega \nabla u) + \underline{(d i_Z \omega) \otimes u} + (-1)^{r-1}(i_Z \omega)\nabla u =$$
$$= \mathcal{L}_Z \omega \otimes u + \underline{(-1)^r (i_Z \omega)\nabla u} + (-1)^{2r} \omega i_Z \nabla u + \underline{(-1)^{r-1}(i_Z \omega)\nabla u} =$$
$$= \mathcal{L}_Z \omega \otimes u + \omega \otimes \nabla_Z u$$

We used the fact that i_Z is an odd derivation, the Cartan magic formula and also the fact that on Ω^0, $\nabla_Z = i_Z \nabla$, since contraction does not even make sense here. So this is the Leibniz rule that ∇_Z obeys. We also have the identity $i_X i_Y + i_Y i_X = 0$ and $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$, from which it follows that

$$\nabla_X i_Y - i_Y \nabla_X = i_{[X,Y]}$$

Note that the identity $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$ can be proved by checking on functions and exact 1-forms, and then inducting. For functions, it is trivial, whereas for exact 1-forms df, we can calculate:

$$\mathcal{L}_X i_Y(df) - i_Y \mathcal{L}_X(df) = (X \circ Y) \cdot f - i_Y(d i_X(df) + i_X d^2 f) =$$
$$= (X \circ Y) \cdot f - i_Y d(X \cdot f) = (X \circ Y) \cdot f - (Y \circ X) \cdot f =$$
$$= [X, Y] \cdot f = i_{[X,Y]}(df)$$

1.8.3 Curvature, globally and locally

Using the machinery from the previous section, we can compute a global version of the curvature:

Formula (Global formula for the curvature):

$$F(X,Y)(u) = i_Y i_X \nabla^2(u) = i_Y(i_X \nabla)\nabla u = i_Y(\nabla_X - \nabla i_X)\nabla u =$$
$$= (i_Y \nabla_X)\nabla u - i_Y \nabla \nabla_X u = (\nabla_X i_Y - i_{[X,Y]})\nabla u - \nabla_Y \nabla_X u =$$
$$= \nabla_X \nabla_Y u - \nabla_Y \nabla_X u - \nabla_{[X,Y]} u$$
$$i.e.$$
$$F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

Locally, the curvature form associated to a connection $\mathcal{A}$ is defined as the End(E)-valued 2-form which locally satisfies $\nabla^2 \sigma = F \wedge \sigma$ and is given by $dA + A \wedge A$. This can be seen by checking locally, using the formula $\nabla \sigma = d\sigma + A \wedge \sigma$. These 2-forms transform using conjugation by the cocycle g. The connection is called flat if $F = 0$.

To compute the curvature locally, we use the local connection one form matrices: from $F(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ we can infer that $F_{ij} = [\nabla_i, \nabla_j]$, since $[\partial_i, \partial_j] = 0$. Recall also that $\nabla^2 = d\Gamma + \Gamma \wedge \Gamma$, where $\nabla = d + \Gamma = d + \Gamma_i dx^i$. (I used A for the matrix-valued one-form, but I like gamma better now, as it is what people tend to use for the Christoffel symbols) Calculating, we get that

$$F_{ij} = (d\Gamma + \Gamma \wedge \Gamma)_{ij} = \partial \Gamma_j / \partial x^i - \partial \Gamma_i / \partial x^j + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i$$

Another way to do this is to use the fact that $\nabla_j e_k = \Gamma^i_{jk} e_i$ and then $F_{ij} = [\nabla_i, \nabla_j]$, so

$$F_{ij} e_k = (\nabla_i \nabla_j - \nabla_j \nabla_i) e_k = \nabla_i (\Gamma_j e_k) - \nabla_j (\Gamma_i e_k) =$$
$$= (\partial_i \Gamma_j - \partial_j \Gamma_i) e_k + (\Gamma^l_{js} \Gamma^k_{il} - \Gamma^l_{is} \Gamma^k_{jl}) e_k = (\partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i) e_k$$

(Notice that $\Gamma^l_{js} \Gamma^k_{il} = \Gamma^k_{il} \Gamma^l_{js}$ is actually a component of $\Gamma_i \Gamma_j$, since the l is common)

> *Remark (Remark):* It is a fact that a Riemannian manifold with flat Levi-Civita connection is locally isometric to $\mathbb{R}^n$ with the usual Euclidean metric. See Week 8.

Example: trivial connection, the connection is just the exterior derivative, we get a chain complex which is the De Rham complex. More generally, any flat connection gives us a chain complex, and also a representation of π_1.

We have the Bianchi identity, which says that the curvature is closed: $d^{\nabla} F = 0$.

> **Proposition 1.14 (First Bianchi identity):** $d^{\nabla} F = 0$

Proof. The curvature is an $End(E)$–valued form, so transforms using the adjoint representation. To see how this new connection acts on the curvature 2-form, recall that $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$. The way the covariant derivative $End(\mathcal{A})$ acts on this is by definition $dF_\alpha + End(\mathcal{A})_\alpha \wedge F_\alpha$. We just saw that $End(\mathcal{A})_\alpha$ is given by the Lie bracket, however we're tensoring with 2-forms so it becomes the commutator with respect to wedging and not multiplication. We get (suppressing indices):

$$d^{End(\mathcal{A})} F = dF + L_A \wedge F = d(dA + A \wedge A) + A \wedge F - F \wedge A =$$
$$= dA \wedge A - A \wedge dA + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A = 0$$

$\square$

This can also be computed purely algebraically using a formula for the induced connection on $End(E)$, which should be given by $\nabla s - s \nabla$. But since $F = \nabla^2$, we see that this vanishes i.e. $\nabla^3 - \nabla^3 = 0$.

For the relationship between curvature and Gaussian curvature - see notes and examples classes recording.

1.8.4 Parallel transport

If we have a bundle E over the unit interval I, then given a starting point (really, a vector) $p \in E_0$, this can be extended uniquely to a horizontal section s with $s(0) = p$. This produces a linear isomorphism $E_0 \to E_1$ sending $s(0) \mapsto s(1)$. For example, when we have a trivial bundle wit trivial connection given by the exterior derivative, the ODE will just be $d = 0$, hence parallel transport will just be constant i.e. the identity map.

The reason for this is that this amounts to solving an ODE with an initial value problem, which by Picard-Lindelöf and compactness of I has a unique global solution. To see this, notice that the equation for the section being horizontal is locally $dv_\alpha + A_\alpha v_\alpha = 0$, where v_α is s in some local trivialization and A_α is a matrix-valued 1-form on I, i.e. looks like $M_\alpha dt$. We get $dv_\alpha/dt\,dt + M_\alpha dt\, v_\alpha = 0$ i.e.

$$\frac{dv_\alpha}{dt} + M_\alpha v_\alpha = 0$$

This is a first-order ODE with an initial value, hence we must be able to find a unique solution. The idea is that for small enough parts of the unit interval, there is a basis of horizontal sections. This is because being a basis is an open condition depending on the determinant, i.e. if it holds at a point, it holds in a small nbhd around the point as well. So split up unit interval in finitely many bits $[a_0, a_1, \ldots, a_n]$, where you pick fibrewise basis and put $s_0 = \sum \lambda_{0j} s^j_)(0)$ and extend this to a horizontal section for a little bit. Then write $s(a_1)$ using s_1^j's and these agree on intersection, and keep going. This produces the desired unique global horizontal section.

In general, for a bundle $E \to X$ and a curve $\gamma : I \to X$, once can just apply the procedure above to the pullback bundle $\gamma^* E$, which amounts to a lifting problem:

$$\begin{tikzcd}
 & E \arrow[d, "\pi"] \\
I \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & B
\end{tikzcd}$$

This defines the parallel transport map

$$P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$$

Notice that this is not a priori homotopy invariant, hence does not produce a representation of $\pi_1(X)$. It does, however, when the curvature is zero, i.e. the connection is flat. In other words, the curvature is the obstruction to this being the case.

The defining equation for parallel transport is that $\tilde{\gamma}$ is horizontal as a section of $\gamma^* E$, i.e.

$$\nabla_{\dot{\gamma}} \tilde{\gamma} = 0$$

Example (Example): Given $\underline{\mathbb{R}^2} \to \mathbb{R}\times S^1$ with local connection one-form $A_\alpha = f\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}dx + g\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}d\theta$ and a curve $\gamma(t) = (t,0) \in \mathbb{R}\times S^1$, we can pull back the connection to get $\gamma^* A_\alpha = f(t,0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}dt$ and hence the ODE for parallel transport becomes $d^{\gamma^* A_\alpha} v = dv/dt\, dt + f\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}v\, dt = 0$, hence $\dot{v} + f(t,0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}v = 0$. The general solution for this is in the form $v = \begin{pmatrix}e^{-\lambda} & 0\\ 0 & e^{\lambda}\end{pmatrix}v_0$, $\lambda = \int_0^t f(x,0)dx$. For a fixed t, we get a matrix, i.e. a linear transformation, which is the holonomy.

Remark (Covariant differentiation is infinitesimal parallel transport): the connection can be recovered from the parallel transport as follows: consider an integral curve c for X. Given a section s, we can take the vector $s(c(t))$ and parallel transport it to $c(0) = x$, giving us an element $\tilde{s}(t) \in E_x$. In other words, for each t, we are taking $s(c(t))$ and parallel transporting it along a curve $\tilde{c}_t$ back to E_x, which should be the value $\tilde{c}_t(-t)$. Combining all times t, we get a collection of curves $\tilde{c}(t,r)$ where for each t, r goes from 0 to $-t$, whose domain is a flipped triangle. We want these to satisfy:

- $\tilde{c}(t,0) = s(c(t))$, i.e. starts at s
- $\nabla_{\dot{c}}\tilde{c}(t,-) = 0$

Then we have that the covariant derivative of s in the direction of X at $c(0)$ is the infinitesimal change in $\tilde{s}$ [a]:

$$\nabla_X(s)(c(0)) = \frac{d}{dt}|_{t=0}\tilde{s}(t) = \frac{d}{dt}|_{t=0}\tilde{c}(t,-t) \tag{1.1}$$

Another way to put this is as saying that

$$P_t\nabla_{\dot{c}} s = \partial_t P_t s,$$

where P_t is parallel transport.

[a] see Minerbe diff geo, 42; or Spivak, chapter 6 prop 3., Ballmann, page 22. They use the alternative approach of using local horizontal sections.

Remark (Continued): In other words, we are taking a section and for each time t, we measure the difference between $s(x)$ and the parallel transport from $s(c(t))$ to E_x, which should properly be a tangent vector in $T_{s(x)}E_x$.
To identify this, we need to look locally and reparametrize $c(t) = t$, and put $E = I \times \mathbb{R}^n$. Then, put $s = (t, S(t)), \tilde{c}(t,r) = (t, C(t,r))$ which we want to differentiate. We know $C(t,0) = S(t)$. Looking locally has the benefit that we can use the local connection matrix and hence the horizontality equation is equivalent to

$$dC(\frac{\partial}{\partial r}) + AC = 0$$

Now, we can differentiate:

$$\frac{d}{dt}|_{t=0}C(t,-t) = [\frac{d}{dt}|_{t=0}C(t,s)]|_{s=0} - [\frac{d}{ds}C(t,s)]|_{t=0} =$$
$$= dC(\partial_t) - dC(\partial_r) = \frac{d}{dt}|_{t=0}S(t) + A_0 C(0,0)$$

On the other hand,

$$(\nabla_{\partial_t}(s))_x = \frac{d}{dt}|_{t=0}S(t) + A_0 S(0)$$

We conclude by noting that $S(0) = C(0,0)$.

This idea of covariant differentiation as infinitesimal parallel transport has interesting geometric interpretations for the curvature and torsion.

Theorem 1.15 (Local interpretation of curvature): *If $\{a,b\}$ denotes the curve corresponding to a small rectangle, then the infinitesimal change in the monodromy (parallel transport) at a point p is*

$$\frac{\partial^2 P_{\{a,b\}}}{\partial a \partial b}|_{a=b=0} = -F_{ij}(p)$$

Proof. To see this, note that $P_{\{a,b\}} = P^{-b} \circ P^{-a} \circ P^b \circ P^a$. Moving in the directions of a small rectangle are integral curves for e.g. ∂_i, ∂_j, then since P^{-a} is parallel transport along this integral cuvre, we have that $\partial P^{-a}/\partial a = \nabla_i$ and similarly $\partial P^a/\partial a = -\nabla_i$. Furthermore, we note that each P is a matrix, and hence the Leibniz rule applies. We compute:

$$\frac{\partial(P^{-b} \circ P^{-a} \circ P^b \circ P^a)}{\partial a}|_{a=0} = P^{-b}\frac{\partial P^{-a}}{\partial a}|_{a=0}P^b P^0 + P^{-b}P^0 P^b \frac{\partial P^a}{\partial a}|_{a=0} = P^{-b}\nabla_i P^b - \nabla_i$$

Now we differentiate w.r.t. b:

$$\frac{P^{-b}\nabla_i P^b - \nabla_i}{\partial b}|_{b=0} = \frac{\partial P^{-b}}{\partial b}\nabla_i P^0 + P^0 \nabla_i \frac{\partial P^b}{\partial b} = \nabla_j \nabla_i - \nabla_i \nabla_j = -[\nabla_i, \nabla_j] = -F_{ij}$$

A different proof is given later, interpreting parallel transport as flowing along horizontal lifts.

□

Remark: parallel transport can be used to prove homotopy invariance of vector bundles.

Example (Example): Given $\underline{\mathbb{R}} \to \mathbb{R}^2$ with local connection one-form $A_\alpha = Cx^1 dx^2$. Of the four curves making up the rectangle, all have holonomy equal to the identity, except $\gamma_2 = (a, bt)$, since for the others either x^1 or x^2 is 0. Note that $\gamma_2^* A = Cab\,dt$. To get the holonomy for this, we must solve the equation $\dot{v} + Cabv = 0$, which has solution $v = e^{-Cabt}$ and hence the holonomy is e^{-Cab}. The theorem then says that $-C$ is minus the curvature, which can also be checked directly: $F = Cdx^1 \wedge dx^2$.
we can pull back the connection to get $\gamma^* A_\alpha = f(t,0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dt$ and hence the ODE for parallel transport becomes $d^{\gamma^* A_\alpha} v = dv/dt\,dt + f\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v\,dt = 0$, hence $\dot{v} + f(t,0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v = 0$. The general solution for this is in the form $v = \begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix} v_0$, $\lambda = \int_0^t f(x,0)dx$. For a fixed t, we get a matrix, i.e. a linear transformation, which is the holonomy.

Example (Induced connection on tangent bundles of spheres): onsider TS^n which we can think of as pairs (x, v) with $v \in \mathbb{R}^{n+1}$ orthogonal to x. The trivial vector bundle over S^n comes equipped with the trivial connection, which locally looks like applying d to all entries. The projection map $\Pi : S^n \times \mathbb{R}^{n+1} \to TS^n$ is given by $(x, v) \mapsto (x, v - xx^T v)$, i.e. a sort of Gram-Schmidt thing where we send $v \mapsto v - x\langle v, x\rangle$. Since x is of unit norm, this is orthogonal to x and lies in TS^n. Now parallel transport is equivalent to lifting a curve $\gamma : I \to S^2$ to a curve $v : I \to TS^2$ such that $v(\dot{t})$ is horizontal, i.e. lies in the complement of the vertical distribution.
A section along a curve γ is horizontal if $\Pi dv = 0$ which in our equation means that $\dot{v} - \gamma\gamma^\perp \dot{v} = 0$. But we can interpret this as saying that $\dot{v}$ is orthogonal to the complement $\gamma^\perp = T_\gamma S^n$, which is just the span of γ.

1.8.5 Connections on the tangent bundle

The connection one forms are written using the matrices $\Gamma^i_{kj} dx^k$. The i-th component of the connection applied to a section in a trivialization is then

$$(\nabla v)^i = dv^i + \Gamma^i_{kj} v^j dx^k$$

For example, $\nabla_{\partial_a} \partial_b = \Gamma^i_{ab} \partial_i$. A word of caution: Jack uses the notation $\Gamma^i_{jk} dx^k$, i.e. with j and k reversed, resulting in everything being flipped around - I wonder why...

Recall that $TX \otimes T^*X \simeq End(TX)$. Hence, the global section in End(TX) which is fibrewise the identity map under this isomorphism gives us a global TX-valued 1-form, called the

Definition 1.16 (Solder form): *Solder form, locally given by $\theta = \partial_{x^i} \otimes dx^i$, in $\Omega^1(TX)$. It evaluates $\theta(X) = X$.*
Torsion: the TX-valued 2-form given by $\nabla\theta \in \Omega^2(TX)$, which is equal to

$$d(\partial_i \otimes dx^i) + (A_\alpha \wedge \partial_i) \otimes dx^i = \Gamma^j_{ik} \partial_j dx^k \wedge dx^i$$

This can also be defined more globally as $T = \nabla_X Y - \nabla_Y X - [X,Y]$. As per the example, we can then evaluate $T(\partial_a, \partial_b) = (\Gamma^i_{ab} - \Gamma^i_{ba})\partial_i$, and this is zero precisely when Γ is symmetric in a, b. It also measures a difference in parallel transport, as will be seen later.

Definition 1.17 (Geodesics): *A geodesic curve γ in X is one such that the curve $\dot\gamma$ in TX is horizontal as a section of $\gamma^* TX$, or $\nabla_{\dot\gamma}\dot\gamma = 0$. In other words, it satisfies the equation*

$$\ddot\gamma^i + (\Gamma^i_{jk} \circ \gamma)\dot\gamma^j \dot\gamma^k = 0$$

In other words, parallel transporting e.g. $\dot\gamma(0)$ along γ will be precisely $\dot\gamma$. Note that we are pulling back dx^k along γ and that's why we get $\dot\gamma^k$.

Here is another way to derive the geodesic equation: put $\gamma = (\gamma^1, \ldots, \gamma^n)$ locally. Then $\dot\gamma = \dot{\gamma^i}\partial_i$ and hence, by using the Leibniz rule and linearity of ∇, we get

$$\nabla_{\dot\gamma}\dot\gamma = \sum_i \nabla_{\dot{\gamma^i}\partial_i}\dot\gamma = \sum_{i,j} \dot\gamma^i \nabla_{\partial_i}(\dot\gamma^j \partial_j) = \sum_{i,j,k} \dot\gamma^i(\partial_i \dot\gamma^j \partial_j + \dot\gamma^j \Gamma^k_{ij}\partial_k)$$

Hence, the ∂_k component is precisely

$$\dot\gamma^i \partial_i \dot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j$$

However, by the chain rule, we also have that $\ddot\gamma^k = \frac{\partial\dot\gamma^k}{\partial\gamma^i}\frac{\partial\gamma^i}{\partial t}$, so the expression is precisely

$$\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0$$

Hence, when the metric is flat i.e. $\Gamma = 0$, we get straight lines.

1.8.6 Horizontal distributions

There is a way to get a canonical vector field associated to a connection ∇. Given a vector field X and an integral curve c with $c(0) = x, \dot c(0) = X$ then we can parallel transport any $v \in E_x$ via a curve $\tilde c$ in E starting at v with $\pi \circ \tilde c = c$ and $\nabla_{\dot c} s = 0$. We can lift X to a vector field on E by setting $\tilde X = \frac{d}{dt}|_{t=0}\tilde c(t) = d\tilde c(\frac{\partial}{\partial t})$, and since $d\tilde c : TI \to TE$, this is a tangent vector in TE. In other words, we take an integral curve, choose a vector, then parallel transport it along the integral curve and then differentiate the lift at the identity, getting a tangent vector in TE.

In particular, the local flow of $\tilde X$ has to be the same as parallel transport along the integral curve for X, almost by definition: the local flow satisfies the ODE $\frac{d}{dt}|_{t=0}\Phi^t = \tilde X$, whereas we define

$\tilde{X} := \frac{d}{dt}|_{t=0}\tilde{c}$, where $\tilde{c}$ is the horizontal lift of c i.e. parallel transport along the integral curve of X. By uniqueness of solutions of ODE's with initial conditions, we see that they must agree.

Definition 1.18 (Horizontal distribution): *The collection of all such tangent vectors is called the horizontal distribution $HE \subset TE$. It is complementary to the vertical distribution $VE = \ker d\pi$. We see that $\tilde{\gamma}$ is a parallel transport lift of γ precisely when $\dot{\tilde{\gamma}} \in HE$. This fits in more with the picture of parallel transport in the setting of principal bundles.*

Remark (Horizontal distribution on a Riemannian manifold): Consider $TM \to M$, where (M, g) is Riemannian manifold. On the one hand, the Levi-Civita connection determines a horizontal subbundle $H^\nabla(TM) \subset T(TM)$. On the other hand, the metric g on M determines the Sasaki metric on TM which produces an orthogonal complement of the horizontal subbundle $\ker d\pi = V(TM) \subset T(TM)$. These coincide!
This is not that surprising if one defines the Sasaki metric using the Levi-Civita connection. Firstly, consider the inclusion and projection composition, which is just a constant map m:

$$\begin{tikzcd} T_mM \arrow[r, "\iota_m"'] \arrow[rr, bend left, "m"] & TM \arrow[r, "\pi"'] & M \end{tikzcd}$$

We can differentiate this, whose composition should be zero:

$$\begin{tikzcd} & T(T_mM) \arrow[r, "d\iota_m"'] \arrow[rr, bend left, "0"] & T(TM) \arrow[r, "d\pi"'] & TM \\ T_mM \arrow[r, "\simeq"] & T_{v_m}(T_mM) \arrow[r] \arrow[u, hook] & T_{v_m}(TM) \arrow[r] \arrow[u, hook] & T_mM \arrow[u, hook] \end{tikzcd}$$

For any $v_m \in T_mM$, we thus have an identification $T_mM \simeq V_{v_m}TM$. We can thus define the Sasaki metric g_T as follows:

- If both vectors are horizontal, calculate the inner product of the associated vectors in T_mM
- If both are vertical w.r.t. Levi-Civita connection, calculate the inner product of their projections under $d\pi$ in T_mM
- If one is horizontal and the other is vertical, declare them to be orthogonal

Hence, $H^\nabla(TM) = V(TM)^\perp$. Note that a curve γ with horizontal lift $\tilde{\gamma}$ obeys $\dot{\tilde{\gamma}} \in H$ or in other words $\dot{\tilde{\gamma}} \perp T_\gamma M$.

Let's see what happens locally: $E|_U \simeq U \times \mathbb{R}^k$, v is now some vector $v_0 \in \mathbb{R}^k$ where U has coords x, and $\mathbb{R}^k$ has coords e, and TE_U has basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial e}$. Reparametrizing $c(t) = t$, the horizontality of $\tilde{c}(t) = (t, C(t)), C : I \to \mathbb{R}^k$ is equivalent to

$$\dot{C} + \Gamma_{\dot{c}} C = 0$$

Hence, the lift $\tilde{X}$ is a vector lying over X which has vector part

$$\frac{d}{dt}|_{t=0}C(t) = -\Gamma_X v_0$$

So, we can write $\tilde{X} = (X, -\Gamma_X v_0)$. When $X = \partial_{x^i}$ and $v_0 = v^a e_a$ we get $\tilde{X} = (\partial_{x^i}, -\Gamma_{\partial_i} e_a v^a)$. [1] Since $\Gamma = \Gamma^n_{km} dx^k$, we see that $\Gamma_{\partial_i} = \Gamma^n_{im}$ as a matrix and hence $\Gamma_{\partial x^i} v_0 = \Gamma^a_{ib} v^b \frac{\partial}{e\, a}$ in the basis e_a, so when we lift to TE the basis e_a should correspond to $\frac{\partial}{\partial s^a}$. Putting it all together, we get:

$$\tilde{X} = \partial_{x^i} - \Gamma^a_{ib} v^b \frac{\partial}{\partial v^a}$$

With this knowledge, let's compute the Lie bracket corresponding to the lifts of ∂_i, ∂_j. We get[2]:

$$[\tilde{\partial_i}, \tilde{\partial_j}] = [\partial_{x^i} - \Gamma^a_{ib} v^b \frac{\partial}{\partial v^a}, \partial_{x^j} - \Gamma^c_{jd} v^d \frac{\partial}{\partial v^c}] = (\partial \Gamma_i / \partial x_j - \partial \Gamma_j / \partial x_i - \Gamma_i \Gamma_j + \Gamma_j \Gamma_i) = -F_{ij}$$

> *Remark (Integrability of the horizontal distribution):* Using the Frobenius theorem (later), this calculation means that the horizontal distribution is integrable if and only if the connection is flat! Also, this gives another interpretation of curvature: given commuting vector fields $[X, Y] = 0$, the curvature is measuring whether their lifts commute: $F(X, Y) = -[\tilde{X}, \tilde{Y}]$.

We can also give a new proof of the geometric interpretation of curvature and torsion: if we think of parallel transport as flows of horizontal lifted vector fields, then

$$\frac{\partial \mathcal{P}_{ts}}{\partial t}|_{t=0} = \frac{\partial(\Phi^{-t}_j \circ \Phi^{-s}_i \circ \Phi^t_j \circ \Phi^s_i)}{\partial t}|_{t=0} = -(\Phi^s_j)^* \tilde{\partial}_i + \tilde{\partial}_i$$

Now we can differentiate w.r.t. s to get the Lie derivative, i.e.

$$-\mathcal{L}_{\tilde{\partial}_j} \tilde{\partial}_i = -[\tilde{\partial}_j, \tilde{\partial}_i] = [\tilde{\partial}_i, \tilde{\partial}_j] = -F_{ij}$$

Similarly, for torsion, define $\gamma_{j,k}(s,t)$ to be the point which starts at a point p, then parallel transports ∂_{x^k} along x_j for time s and then flows for time t alond this transported vector. We recall that parallel transport for x_j is the same as the flow of $\tilde{\partial}_{x^j}$, which we denote $\tilde{\Phi}^s_j$. Similarly, if Φ^t_X denotes the flow of a vector field X, we get

$$\gamma_{j,k}(s,t) = \Phi^t_{\tilde{\Phi}^s(\partial_{x^k}(p))}(p + s x_j)$$

When we differentiate w.r.t. t, we get exactly $\tilde{\Phi}^s_j(\partial_{x^k}(p))$, which is the parallel transport of ∂_k along x_j and now we recall that infinitesimal parallel transport is covariant differentiation (but note we have switched directions) to conclude:

$$\frac{\partial^2 \gamma_{j,k}(s,t)}{\partial s \partial t}|_{t=s=0} = -\nabla_{\partial_{x^j}} \partial_{x^k}$$

[1] But now be careful - on one hand, we have thought of $\dot{s}$ as a vector in $\mathbb{R}^k$ but now we're thinking of it as a tangent vector in the basis $\partial / \partial y^a$

[2] The first two terms are obvious. The others are as follows: we get $\Gamma^n_{im} y^m \frac{\partial}{\partial y^n}(\Gamma^l_{jr} y^r \frac{\partial}{\partial y^l}) = stuff + \Gamma^n_{im} \Gamma^l_{jn} y^m \frac{\partial}{\partial y^l}$. The stuff cancels with the other stuff. Hence, from this we get $(\Gamma_i)^n_m (\Gamma_j)^l_n = (\Gamma^T_i)^m_n (\Gamma^T_j)^n_l = (\Gamma^T_i \Gamma^T_j)^m_l = (\Gamma_j \Gamma_i)^l_m$ term for $y^m \frac{\partial}{\partial y^l}$, and similarly for the other one.

We can thus reinterpret the torsion as

$$T(\partial_{x^j}, \partial_{x^k}) = \frac{\partial^2(\gamma_{k,j} - \gamma_{j,k})}{\partial s \partial t}|_{t=s=0}$$

Finally, note that if the connection is flat, the holonomy depends only on the homotopy class of paths, and then we have a representation of $\pi_1(B)$ in $\mathrm{GL}(E_x) \simeq \mathrm{GL}_{\dim E}(\mathbb{R})$. Conversely, given such a representation ρ, we may define $E = \tilde{B} \times \mathbb{R}^k / \sim$, where we mod out by the action of $G = \pi_1(B)$ which is the deck transformation group of the universal cover $\tilde{B}$, i.e. $(b, v) \sim (g \cdot b, \rho(g) \cdot v)$. The point is that $\tilde{B} \times \mathbb{R}^k \to \tilde{B}$ has a trivial flat connection which is invariant under G, hence it descends to a flat connection on E.

1.9 Lie derivatives

Given a vector field $v : X \to TX$, an integral curve starting at p is a curve making the following diagram commute:

$$\begin{tikzcd} TI \arrow[r, "d\gamma"] & TX \\ I \arrow[u, "\partial/\partial t"] \arrow[r, "\gamma"'] & X \arrow[u, "v"'] \end{tikzcd}$$

In other words, solves the ODE $\gamma'(t) = v_{\gamma(t)}$, $\gamma(0) = p$.

By standard ODE theory, integral curves for vector fields exist and when X is compact, they are global and unique. This defines an action of $\mathbb{R}$ on X defined by the flow $\Phi(t,x)$ - the additivity is seen by showing that both $\Phi(t+s,-)$ and $\Phi(t,-)\circ\Phi(s,-)$ solve the same ODE.

Definition 1.19 (Lie derivative): *The Lie derivative of a tensor T in the direction of v is:*

$$\mathcal{L}_v T = \frac{d}{dt}|_{t=0}(\Phi^t)^* T$$

In other words, it measures the ininitesimal change of T along the flow of v.

We have that $(\Phi^t)^*\mathcal{L}_v T = \frac{d}{dt}(\Phi^t)^* T$ and also $\mathcal{L}_v f = df(v) = v(f)$. For one-forms, we can use the chain rule and the fact that d commutes with d/dt to get:

$$\mathcal{L}_v(\alpha) = \frac{d}{dt}|_{t=0}(\alpha_i \circ \Phi^t) d(x^i \circ \Phi^t) = \mathcal{L}_v(\alpha_i) dx^i + \alpha_i d\mathcal{L}_v(x^i) = v^j \frac{\partial \alpha_i}{\partial x^j} dx^i + \alpha_i dv^i$$

It also satisfies a Leibniz rule for forms and vector fields X:

$$\mathcal{L}_v(\alpha(X)) = (\mathcal{L}_v\alpha)(X) + \alpha(\mathcal{L}_v(X))$$

It also satisfies a more general Leibniz rule, since it is defined via a derivative.

Proposition 1.20 (The Cartan magic formula):

$$\mathcal{L}_X\omega = i_X d\omega + d i_X \omega$$

Proof. In [here](obsidian://open?vault=Obsidian □

As a corollary, we get that the Lie derivative commutes with the exterior derivative.

Definition 1.21 (Lie bracket): *The Lie bracket of two vector fields is defined to be* $[v,w] := \mathcal{L}_v w$

In coordinates, one can show that this is equal to the expression

$$(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}) \frac{\partial}{\partial x^j}$$

Proposition 1.22 (Differential is Lie algebra homomorphism): *Given* $M \xrightarrow{F} N \xrightarrow{f} \mathbb{R}$, *then*

$$F_*(\mathcal{L}_X Y) = F_*[X,Y] = [F_*X, F_*Y] = \mathcal{L}_{F_*X}(F_*Y)$$

Proof. Let's say X is F -related to X' and Y to Y'. Then $X\cdot(f\circ F) = (X'f)\circ F$ and similarly with Y. Then,

$$X\cdot Y\cdot(f\circ F) = X\cdot(Y'f\circ F) = (X'Y'f)\circ F$$

Hence,

$$[X,Y](f\circ F) = (X'Y'f - Y'X'f)\circ F = ([X',Y']f)\circ F$$

□

Proposition 1.23 (Homotopy invariance of De Rham cohomology): *De Rham cohomology is a homotopy invariant*

Proof. We will use Cartan's magic formula. Firstly, let $F: I\times X \to Y$ be the homotopy, and define $i_t : X \to I\times X, x \mapsto (t,x)$. Note that $i_t = \Phi^t \circ i_0$, where Φ is the flow of ∂_t, i.e. translation in t direction. Now:

$$F_1^*\alpha - F_0^*\alpha = \int_0^1 \frac{d}{dt} F_t^* dt = \int_0^1 \frac{d}{dt} i_0^*(\Phi^t)^* F^*\alpha \, dt =$$
$$= \int_0^1 i_0^* \frac{d}{dt}(\Phi^t)^* F^*\alpha \, dt = \int_0^1 i_0^*(\Phi^t)^* \mathcal{L}_{\partial_t}(F^*\alpha)\, dt = \int_0^1 i_t^* \mathcal{L}_{\partial_t}(F^*\alpha)\, dt$$

The last bit follows by the property right after the definition of the Lie derivative. When α is closed, the magic formula tells us that $\mathcal{L}_{\partial_t}(F^*\alpha) = d(\iota_{\partial_t} F^*\alpha)$ is exact, and hence we can pull out the differential: $F_1^*\alpha - F_0^*\alpha = d\int_0^1 i_t^* \iota_{\partial_t} F^*\alpha \, dt$ □

1.10 Foliations and integrability

Some examples: fibers of a submersion form a foliation of the source manifold; cosets of subgroups of Lie groups, with the canonical example being $\mathbb{R}^p \subset \mathbb{R}^n$. A k-foliated atlas is one where the transition functions send $x, y \mapsto \zeta(x,y), \eta(y)$, i.e. the slices correspond to y=const. This allows the level sets to be well-defined.

The definition of integrability we are using is that it can be written as $\langle \partial_{x_1}, ..., \partial_{x_k} \rangle$ for k-foliation coordinates $x_1, ..., x_k, y_1, ..., y_{n-k}$.

> **Theorem 1.24 (Frobenius integrability):** *A distribution is integrable if and only if it is closed under the Lie bracket.*

One side of this is obvious. Conversely, assume that D is closed under the Lie bracket. Choose local coordinates about p $s^1, ..., s^k, t^1, ..., t^{n-k}$ such that at p we have that $D = \langle \partial_{s_1}, ..., \partial_{s_k} \rangle$ and $p = 0$. (we can do this, I think it's proven somewhere in Lee). Now around p, we can correct this to ensure that

$$v^i := \partial_{s^i} + \sum a_{ij} \partial_{t^j} \in D$$

for some (unique) smooth functions a_{ij}. Then D is spanned by the v^i and we want to show that there are some nice local k-foliation coordinates inducing them.

Denote Φ_i the local flow of v^i. Define a parametrisation

$$F : (x,y) \mapsto \Phi_1^{x_1} \circ ... \circ \Phi_k^{x_k}(s = 0, t = y)$$

Since p is 0 in these coords, this gives a map from an open nbhd of 0 to an open nbhd of p. At p, we can check that $D_0F(\partial_{x^i}) = v_i(p) = \partial_{s^i}$ (by definition of flows), and $D_0F(\partial_{y^j}) = \partial_{t^j}$ since the y_j are unaffected by the flows. This means that F is a local diffeo by the Inverse Function theorem, and gives a parametrization. The main idea is to use the fact that D is closed under the Lie bracket to show that the flows commute and that this holds not only at p but about p as well.

But by the closure of the Lie bracket, $[v_i, v_j] = \sum b_{ijl} v_l$. By equating the coefficients of ∂_{s^i}, we see that all the b's vanish and hence that the flows commute. Hence,

$$DF(\partial_{x^i}) = \partial F / \partial x^i = \frac{d}{dt}|_{t=0} \Phi_1^{x^1} \circ ... \circ \Phi_i^{x^i + t} \circ \circ \Phi_k^{x^k}(0,y) = \frac{d}{dt}|_{t=0} \Phi_i^{x^i+t} \circ ... \circ \Phi_k^{x^k}(0,y) = v_i$$

and the relation holds around p as well. This proves the theorem.

There's some handwaving here I guess - the stuff about the local coordinates in particular.

Another way to say this is that a p-dimensional distribution D is involutive iff locally there are coordinates $x^1, ..., x^n$ such that $D = \langle \frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^p} \rangle$.

Example (Example): Given a 1-form α, we can ask whether it is locally exact, a la the Poincare lemma. It turns out that the Frobenius theorem can answer this! Consider the distribution on $M \times \mathbb{R}$

$$D = (\xi, \alpha(\xi))$$

where $\xi \in T_x M$. An integral submanifold is a submanifold tangent to this distribution. But D is always transverse to the $\mathbb{R}$ factor of $T(M \times \mathbb{R})$, so such an integral submanifold can be seen as the graph of a function $f : M \to \mathbb{R}$ with tangent bundle (see [[Tangent space to graph]] given by:

$$(\xi, df(\xi))$$

Hence, D is integrable if and only if $\alpha = df$ locally. If $\alpha = \alpha_i dx^i$, then D is generated by $X_i = \frac{\partial}{\partial x^i} + \alpha_i \frac{\partial}{\partial t}$ with commutators being equal to

$$[X_i, X_j] = (\partial_i \alpha_j - \partial_j \alpha_i)\partial_t$$

All in all D is involutive if and only if $\partial_i \alpha_j - \partial_j \alpha_i = 0$, which is the same as $d\alpha = 0$.

Proposition 1.25 (Dual distribution algebra): *A k-plane distribution D is integrable if and only if the annihilator*

$$I(D) = \{\alpha \in \Omega^\bullet(X) | \alpha(v_1, \ldots, v_r) = 0 \, \forall v_i \in D\}$$

is closed under the exterior derivative.

Proof. Look locally on U, where the distribution is $\langle v_1, \ldots, v_k \rangle$. Taking $\alpha_1, \ldots, \alpha_{n-k}$ to be the complement of the dual 1-forms, we see that on U,

$$I(D)|_U = \bigoplus_1^{n-k} \Omega^\bullet(U) \wedge \alpha_i$$

But now integrabitliy if the same as closure under the Lie bracket, so

$$[v_i, v_j] = 0 \iff \alpha_l [v_i, v_l] = 0$$

for all α_l. But we now apply the Leibniz rule to see that

$$0 = \mathcal{L}_{v_i}(\alpha_l(v_j)) = (\mathcal{L}_{v_i}\alpha_l)(v_j) + \alpha_l(\mathcal{L}_{v_i} v_j) = (d\iota_{v_i}\alpha_l + \iota_{v_i} d\alpha_l)(v_j) + \alpha_l([v_i, v_j])$$

and hence the forms vanish on the Lie bracket precisely when they are closed under d. (Note that this is a special case of a more general coordinate free description of the exterior derivative) $\square$

1.11 Riemannian Geometry and the Levi-Civita connection

A Riemannian manifold is a manifold equipped with an inner product g on its tangent bundle.

Proposition 1.26 (Zero torsion formula for exterior derivative): *If a connection ∇ has zero torsion, then*

$$(d\alpha)(v_0,...,v_k) = \sum(-1)^{i+1}(\nabla_{v_i}\alpha)(v_1,...,\hat{v}_i,...,v_{k+1})$$

Proof. A connection on the tangent bundle induces a dual connection on the cotangent bundle and on its exterior powers by the Leibniz formula

$$d(\alpha(w_1,...,w_k)) = (\nabla\alpha)(w_1,...,w_k) + \alpha(\nabla w_1,...,w_k) + ... + \alpha(w_1,...,\nabla w_k)$$

Now recall the formula

$$(d\alpha)(v_1,...,v_{k+1}) = \sum(-1)^{i+1}v_i \cdot \alpha(v_0,...,\hat{v}_i,...,v_k) + \sum_{i<j}(-1)^{i+j}\alpha([v_i,v_j],...,v_{k+1})$$

But now

$$\sum(-1)^{i+1}(\nabla_{v_i}\alpha)(v_1,...,\hat{v}_i,...,v_{k+1}) = \sum(-1)^{i+1}[(v_i \cdot \alpha(v_0,...,\hat{v}_i,...,v_k) - \alpha(\nabla_{v_i}v_0,...,\hat{v}_i,...,v_k) - ... - \alpha(v_0,...,\hat{v}_i,...,\nabla_{v_i}v_k)]$$

Now, the first terms is the same. On the other hand, for the second one, each pair (i,j) with $i<j$ is going to appear either as $(-1)^i\alpha(...\nabla_{v_iv_j}...)$ or as $(-1)^j\alpha(...\nabla_{v_j}v_i...)$, which after moving them to first position will result in

$$(-1)^{i+j}\alpha(\nabla_{v_i}v_j - \nabla_{v_j}v_i,...) = (-1)^{i+j}\alpha([v_i,v_j],...)$$

and we're done. □

1.11.1 The Levi-Civita connection

Proposition 1.27 (Levi-Civitia connection): *The Levi-Civita connection is the unique connection on TM which is torsion-free and compatible with the metric. On $\mathbb{R}^\ltimes$ this is the trivial connection, and on a submanifold of $\mathbb{R}^n$, it is the induced pullback by the inclusion map of the trivial connection.*

Proof. The proof proceeds by showing there is a bijection:

$$\{\text{orthogonal connections on TM}\} \leftrightarrow \Omega^2(TM)$$
$$\nabla \mapsto T_\nabla$$

Fixing any orthogonal connection $\mathcal{A}$, then any other orthogonal connection differs by some Δ, a $\mathfrak{o}(TX)$-valued 1-form. Hence, we actually consider the linear map $\Delta \mapsto T_{\mathcal{A}+\Delta} - T_{\mathcal{A}}$, given by wedging with θ, the solder form. In other words, we have a map of bundles

$$\mathfrak{o}(TX)\otimes T^*X \to TX \otimes \Lambda^2 T^*X$$

Both of these bundles have rank $n\binom{n}{2}$, hence to show theyre isomorphic we need only show wedging with θ is injective. But $\Delta \wedge \theta = \Delta^i_{kj} - \Delta^i_{jk}$. So if Δ gets mapped to 0, we must first have that $\Delta_{ikj} = \Delta_{ijk}$ and also that $\Delta_{ijk} = -\Delta_{jik}$, since we're assuming it is in $\mathfrak{o}(TX)$. All in all, we have that

$$\Delta_{ijk} = -\Delta_{jik} = -\Delta_{jki} = \Delta_{kji} = \Delta_{kij} = -\Delta_{ikj} = -\Delta_{ijk}$$

and so $\Delta = 0$. □

Now, to actually calculate a formula for the Christoffel symbols of the Levi-Civita connection, we need to do some work. Recall that the defining properties of zero torsion and compatibility with the metric are:

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$$

In particular, putting $s_1 = X, s_2 = Y$ and contracting with Z, we get the equation:

$$i_Z d\langle X,Y\rangle = Z \circ \langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

Permuting the X,Y and Z, we can sum to get:

$$\begin{gathered} Z\circ\langle X,Y\rangle - Y\circ\langle X,Z\rangle + X\circ\langle Y,Z\rangle = \\ = \langle \nabla_Z X, Y\rangle + \langle X,\nabla_Z Y\rangle - \langle \nabla_Y X, Z\rangle - \langle X, \nabla_Y Z\rangle + \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \\ = \langle \nabla_X Y - \nabla_Y X, Z\rangle + \langle \nabla_Z X - \nabla_X Z + 2\nabla_X Z, Y\rangle + \langle \nabla_Z Y - \nabla_Y Z, X\rangle = \\ = \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle + 2\langle \nabla_X Z, Y\rangle \end{gathered}$$

All in all, we deduce that:

$$\begin{gathered} \langle \nabla_X Z, Y\rangle = \frac{1}{2}(Z\circ\langle X,Y\rangle - Y\circ\langle X,Z\rangle + X\circ\langle Y,Z\rangle \\ -\langle [X,Y],Z\rangle - \langle [Z,X],Y\rangle + \langle [Y,Z],X\rangle) \end{gathered}$$

This is called the Koszul formula. Plugging in $X = \partial_i, Y = \partial_k, Z = \partial_j$, we get firstly that $\nabla_X Z = \nabla_{\partial_i}\partial_j = \Gamma^l_{ij}\partial_l$. Now,

$$\langle \nabla_{\partial_i}\partial_j, \partial_k\rangle = \langle \Gamma^l_{ij}\partial_l, \partial_k\rangle = \Gamma^l_{ij} g_{kl}$$

On the other hand, the Koszul formula tells us, since all of X,Y and Z commute, that the right hand side is

$$\frac{1}{2}(\partial_j g_{ik} - \partial_k g_{ij} + \partial_i g_{kj})$$

If we denote by g^{kl} to be the inverse of g_{kl}, we arrive at the formula:

Formula (Formula):

$$\Gamma^l_{ij} = \frac{1}{2}\sum_k g^{kl}(\partial_i g_{kj} - \partial_k g_{ij} + \partial_j g_{ik})$$

1.11.2 Geodesics for the Levi-Civita connection

From this formula, we can infer that geodesics are locally action minimizing. Let $F : TM \to \mathbb{R}$ be the norm function, i.e. $F(x,v) = \langle v,v\rangle = g_{jk}v^j v^k$. Given a curve $\gamma : \mathbb{R} \to M$, we can lift it to $\tilde{\gamma} : \mathbb{R} \to TM, t \mapsto (\gamma, \dot{\gamma})$. Its action/length is then defined as

$$\mathcal{A}(\gamma) = \int_a^b \tilde{\gamma}^* F\, dt = \int_a^b |\dot{\gamma}|^2\, dt$$

For a curve to minimize this action, it has to satisfy the Euler-Lagrange equations:

$$\frac{\partial F}{\partial x^i}(\gamma, \dot{\gamma}) = \frac{d}{dt}\frac{\partial F}{\partial v^i}(\gamma, \dot{\gamma})$$

The left hand side of this equation is

$$\sum_{j,k} \partial_i g_{jk} \dot{\gamma}^j \dot{\gamma}^k$$

On the other hand, the right hand side is, by the Leibniz rule and chain rule,

$$\begin{aligned}\frac{d}{dt}(g_{ik}v^k + g_{ji}v^j)(\gamma, \dot{\gamma}) &= \partial_l g_{ik}(\gamma)\frac{\partial \gamma^l}{\partial t}\dot{\gamma}^k + g_{ik}\ddot{\gamma}^k + \partial_l g_{ji}(\gamma)\frac{\partial \gamma^l}{\partial t}\dot{\gamma}^j + g_{ji}\ddot{\gamma}^j = \\ &= (\partial_j g_{ik} + \partial_k g_{ji})\dot{\gamma}^j \dot{\gamma}^k + 2g_{il}\ddot{\gamma}^l\end{aligned}$$

Hence, we can rewrite the Euler-Lagrange equation as:

$$\ddot{\gamma}^l + \frac{1}{2}g^{il}(\partial_j g_{ik} + \partial_k g_{ji} - \partial_i g_{jk})\dot{\gamma}^j\dot{\gamma}^k = \ddot{\gamma}^l + (\Gamma^l_{jk} \circ \gamma)\dot{\gamma}^j\dot{\gamma}^k = 0$$

which is precisely the geodesic equation! Hence, geodesics for the Levi-Civita connection are locally action minimizing (and also locally length minimizing).

1.11.3 Riemannian curvature

Fix a Riemannian manifold (M, g). Then, its curvature with respect to the Levi-Civita connection defines the curvature tensor R:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$R(\partial_j, \partial_k)\partial_i = R^l_{ijk}\partial_l$$

If we apply the Leibniz rule, we actually see that:

$$\nabla_j \nabla_k \partial_i = \nabla_j(\Gamma^s_{ki}\partial_s) = (\partial_j \Gamma^s_{ki})\partial_s + \Gamma^s_{ki}\nabla_j\partial_s = (\partial_j\Gamma^l_{ki} + \Gamma^s_{ki}\Gamma^l_{js})\partial_l = (\partial_j \Gamma^l_{ik} + \Gamma^l_{sj}\Gamma^s_{ik})\partial_l$$

Similarly,

$$\nabla_k \nabla_j \partial_i = (\partial_k \Gamma^l_{ij} + \Gamma^l_{sk}\Gamma^s_{ij})\partial_l$$

All in all,

Formula (Formula for the Riemann tensor):

$$R^l_{ijk} = \partial_j \Gamma^l_{ik} + \Gamma^l_{sj}\Gamma^s_{ik} - (\partial_k \Gamma^l_{ij} + \Gamma^l_{sk}\Gamma^s_{ij}) = \partial_j \Gamma^l_{ik} - \partial_k \Gamma^l_{ij} + \Gamma^l_{sj}\Gamma^s_{ik} - \Gamma^l_{sk}\Gamma^s_{ij}$$

We see that since this is a 2-form and is skew-symmetric, then $R_{ijkl} = -R_{jikl} = -R_{ijlk}$.

There is also a third formula, due to the first Bianchi identity:

Formula (Jacobi-style identity):

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Proof. Writing everything out, we get:

$$\underline{\nabla_X \nabla_Y Z} - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \underline{\nabla_{[Y,Z]} X} +$$
$$\nabla_Z \nabla_X Y - \underline{\nabla_X \nabla_Z Y} - \nabla_{[Z,X]} Y$$

The underlined terms evaluate, given torsion-freeness, to $[X,[Y,Z]]$. Similarly, so do the other terms and all in all we get:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

due to the Jacobi identity. So this is really the Jacobi identity in disguise! □

Knowing a bunch of properties of Riemannian curvature, we are ready to prove a fundamental theorem:

Theorem 1.28 (Theorem): *The Levi-Civita connection of a Riemannian manifold (M,g) is flat, i.e. has $R = 0$ if and only it is locally isometric to Euclidean space with the usual metric.*

Proof. If the connection is flat, there exists, around any point, an orthonormal frame of horizontal vector fields X_i. This is because the flatness of the connection ensures that the horizontal distribution is integrable. Now, by torsion freeness, we have that

$$[X_i, X_j] = \nabla_i X_j - \nabla_j X_i = 0$$

Hence, there exist local coordinates x^i with $X_i = \frac{\partial}{\partial x^i}$ and hence $g_{ij} = \delta_{ij}$. □

1.12 Hodge Theory

The metric g induces an inner product on all p-forms, with α^I local, orthonormal fibrewise basis. The Hodge star is defined as the unique $n-p$-form such that the following holds:

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega$$

Its square is $** = (-1)^{p(n-p)}$. This lets us define another inner product on forms as follows:

$$\langle \alpha, \beta \rangle_X := \int_X \alpha \wedge *\beta = \int_X \langle \alpha, \beta \rangle \omega$$

Using Stokes' theorem, one can show that d has the formal adjoint $d^* = (-1)^{np+n+1} * d *$. The Laplace operator is defined as

$$\Delta := dd^* + d^*d$$

One can then show that the harmonic forms satisfy $\Delta\alpha = 0 \iff d\alpha = d^*\alpha = 0$, and that there is a bijection between de Rham classes and harmonic forms: $\mathcal{H}^p \leftrightarrow H^p$.

1.13 Principal bundles

A Lie group is a manifold with a compatible smooth group strucure. It comes with maps L_g, R_g, C_g for all $g \in G$. Given this, we can define the notion of a left-invariant vector field as one such that $dL_g v = v$, i.e. the following commutes:

It turns out that these are in bijection with the tangent vectors at e, i.e. $LIVF's \leftrightarrow T_eG = \mathfrak{g}$, the Lie algebra of the Lie group G. This defines a Lie bracket on $\mathfrak{g}$, by translating the usual Lie bracket of vector fields using the bijection.

The conjugation action of G on itself turns into an action of G on its Lie algebra: $Ad_g(\xi) = g \cdot \xi = dC_g \xi$.

Definition 1.29 (Infinitesimal action): *Given a Lie group action $\sigma : G \times X \to X$, we get an associated infinitesimal Lie algebra action $\mathfrak{g} \to \mathfrak{X}(X)$ by differentiating. It can be defined as follows, where γ is a curve representing ξ:*

$$\xi \cdot x = d\sigma_{(e,x)}(\xi, 0) = [\gamma(t) \cdot_\sigma x] : T_eG \times T_xX \to T_xX$$

Given a left-invariant vector field $V \in \mathfrak{g}$, we can consider its flow i.e. the exponential map $t \mapsto \exp(tV)$, giving rise to a curve $\exp(TV) \cdot x$ for all $x \in X$. Its derivative at 0 is a tangent vector in X, which is the same thing defined above. One can also think of σ as giving a map $G \to Diff(X)$, where $Diff(X)$ is an infinite dimensional manifold. A flow can be thought of as a path in $Diff(X)$ passing through the identity map. The differential of this map then gives the desired map $\mathfrak{g} \to \mathfrak{X}(X)$, and this turns out to be a Lie algebra homomorphism.

Another way to describe this is, given any $x \in X$, to consider the map $\sigma_x : G \to X, g \mapsto g \cdot x$. The derivative of this map is then $d\sigma_x : TG \to TX$

Definition 1.30 (Principal bundle): *A principal bundle is a space $P \to B$ with fibers diffeomorphic to a fixed Lie group G, such that there is a cover for B by U's' with trivializations $P_U \simeq U \times G$ and transition functions $U \cap V \to G$.*

To every vector bundle $E \to B$, we can associate a principal $GL_k(\mathbb{R})$-bundle, called the frame bundle, as follows: $F(E)_b = \{\text{ordered bases of } E_b\}$. The transition functions are the same as the transition functions for E.

There is a right action on fibers given by $(b,x) \cdot g = (b, xg)$ in trivializations. This makes sense in different trivializations, because the change of coordinates acts by multiplying on the left by an element $g_{\beta\alpha}(b) \in G$, which commmutes with multiplication on the right.

Given this right action, we can consider the construction from the previous section. Namely, we have a map $i_p : G \to P, g \mapsto p \cdot g$. The image of this map consists of the whole fiber that p is in.

Differentiating, we get a map

$$di_p : T_eG \to T_pP$$

and furthermore a short exact sequence:

$$0 \to T_eG \to T_pP \to T_xB \to 0$$

An Ehresmann connection is a splitting of this sequence, with the image of $\mathfrak{g}$ consisting of vertical vectors and its complement being the horizontal vectors. We can define the fundamental vector field, for $A \in \mathfrak{g}$ by:

$$A^\flat(p) = di_p(A) = \frac{d}{dt}|_{t=0}\, i_p(\exp(tA)) = \frac{d}{dt}|_{t=0}\, p \cdot \exp(tA) \in Vert(T_pP)$$

This is a vector field $P \to TP$, and in fact this operation gives a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(P), A \mapsto A^\flat$. To see why, check [here](obsidian://open?vault=Obsidian

Definition 1.31 (Connection on a principal bundle): *A connection on a principal bundle $P \to B$ is a $\mathfrak{g}$-valued 1-form $\omega : TP \to \mathfrak{g}$ obeying the following:*

- $\omega_p(A^\flat(p)) = A$, *i.e.* $P \xrightarrow{A^\flat} TP \xrightarrow{\omega} \mathfrak{g}$ *composes to the constant map A, fixes the vertical vectors.*
- *Moreover,* $R_g^*\omega = Ad_{g^{-1}}\omega$.

The kernel of this 1-form consists of the horizontal vectors. The equivariance condition is the commutativity of the following diagram:

Connections on a vector bundle E turn out to be equivalent to connections on the frame bundle $F(E)$. The local connection one-forms obey the following transformation rule:

$$\mathcal{A} = Ad_{g^{-1}}\mathcal{A}' + dL_{g^{-1}}dg$$

Definition 1.32 (Curvature on principal bundles): *The curvature of $\mathcal{A}$ is the $\mathfrak{g}$-valued 2-form $\mathcal{F}$ defined by $d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$, where $[(\xi_i \otimes \sigma_i) \wedge (\eta_j \otimes \tau_j)] = [\xi_i, \eta_j] \otimes (\sigma_i \wedge \tau_j)$.*

Again, the horizontal distribution is integrable if and only if the curvature is 0, i.e. the connection is flat.

Here is another way to think of a connection on a principal bundle. Given $\pi : P \to M$, there is a short exact sequence of bundles

$$0 \to \ker(d\pi) \to TP \to \pi^*TM \to 0$$

The subbundle $\ker(d\pi)$ is called the vertical subbundle and consists of vectors tangent to the fibers of P. Now, the right action of G on P can be lifted to an action on TP by differentiating, i.e. if $m_g : P \to P, p \mapsto p \cdot g$, then $dm_g : TP \to TP$, and this preserves the vertical subbundle, since multiplying on the right by an element of G stays in the same fiber, i.e. $\pi \circ m_g = \pi$.

The action of G on π^*TM is defined as the induced action on it as a subbundle of $P \times TM$, where G acts only on the first factor, i.e. $(p, v) \mapsto (pg^{-1}, v)$. With this in mind, the SES becomes an SES of vector bundles equipped with an action of G.

Moreover, it is true that $\ker(d\pi) \simeq P \times \mathfrak{g}$ for the following reason: there is a map $\psi : P \times \mathfrak{g} \to \ker(d\pi)$ that takes a pair (p, ξ) and outputs the tangent vector at $t = 0$ represented by the curve $\gamma(t) = p \cdot \exp(t\xi)$. Since this curve actually lands in the fiber of p, its composition with π is constant and hence $d\pi(\gamma'(0)) = (\pi \circ \gamma)'(0) = 0$, i.e. this vector is in $\ker(d\pi)$. This is moreover G-equivariant, with the adjoint action on $\mathfrak{g}$, since

$$\psi(pg^{-1}, gvg^{-1}) = dm_g \psi(p, v)$$

To see why, the tangent vector $\psi(pg^{-1}, gvg^{-1})$ is represented by the curve $t \mapsto p \cdot g^{-1}\exp(tgvg^{-1})$, which is the same as $t \mapsto p \cdot \exp(tvg^{-1})$. However, $dm_g\psi(p,v)$ is represented by the composition of m_g with γ, which is precisely the curve $t \mapsto p \cdot \exp(\mathrm{tv})g^{-1}$, and these are the same.

Now, we can redefine a connection on P as an equivariant splitting of the SES, in other words a map

$$A : TP \to \ker(d\pi)$$

which is a retraction, i.e. is the identity on the vertical subbundle, and is G-equivariant: $dm_g(Av) = A(dm_g v)$. Using the isomorphism $\ker(d\pi) \simeq P \times \mathfrak{g}$, we can rephrase this as follows: we have a $\mathfrak{g}$-valued 1-form on P, i.e. a map

$$\omega_A : TP \to \mathfrak{g}$$

with the property that - $\omega_A(\psi(p, v)) = v$, since the previous definition required it to be a retraction, i.e. to fix the vertical vectors - $\omega_A(dm_g v) = R^*_{g^{-1}}\omega_A(v) = g\omega_A(v)g^{-1} = Ad_g\omega_A$, i.e. G-equivariance rephrased in this new setting.

This is the same definition as before!

1.13.1 Parallel transport on principal bundles

Given a curve $\gamma : I \to M$ and a point $p_0 \in P$, there is a unique lift $\tilde{\gamma} : I \to P$ starting at p_0 and whose derivative is a horizontal tangent vector for all t.

This, again, comes from solving an ODE of the form $dv + Av = 0$ along the curve. Using parallel transport, we can also redefine covariant derivatives as infinitesimal parallel transport, i.e.

$$\nabla_X s(x_0) = \partial_t P_t(s)|_{t=0} = \lim_{t \to 0} \frac{P^{\gamma}_{-t} s(\gamma(t)) - s(x_0)}{t}$$

2 Complex Manifolds

2.1 Local theory

2.1.1 Classic theorems from complex analysis

Theorem 2.1 (Multivariable CIF):

$$f(z) = \frac{1}{2\pi i}^n \int_{|w_j - a_j| = r_j} \frac{f(w)}{(w_1 - z_1)\ldots(w_n - z_n)} dw$$

Maximum principle, Identity principle.

2.1.2 Extension theorems

Firstly, an important difference between classical complex analysis in one variable and the multivariable case comes in the form of the extension theorems:

Theorem 2.2 (Hartogs' theorem): *A holomorphic map $f : B_\epsilon \backslash \overline{B_{\epsilon'}} \to \mathbb{C}$ can be holomorphically extended uniquely to B_ϵ, if $n > 1$.*

Proof. $f(z_1, \ldots, z_n)$ can be thought of as a holomorphic function of $w = (z_2, \ldots, z_n)$ via $z_1 \mapsto f(z_1, \ldots, z_n)$ for fixed z_1, but also as a holo function of z_1 via $f_w(z_1)$. $\square$

Theorem 2.3 (Riemann extension theorems): *Holomorphic $f : U \setminus Z(f) \to \mathbb{C}$ locally bounded near the zero set extend to $\tilde{f} : U \to \mathbb{C}$. If Y is a codimension at least two subset of U, then we can remove the boundedness assumption and show that the restriction map $\mathcal{O}(U) \to \mathcal{O}(U \setminus Y)$ is bijective, i.e. holo functions $U \setminus Y \to \mathbb{C}$ extend to U.*

Proof. See here: `https://link.springer.com/content/pdf/10.1007/978-3-642-69582-7_7?pdf=chapter%20toc` $\square$

2.1.3 Weierstrass division and local rings

A Weierstrass polynomial is a polynomial of the form

$$z^d + \alpha_1(w) z^{d-1} + \ldots$$

where α_i is a holo function in $n-1$ variables. We have the following important theorem:

Theorem 2.4 (Weierstrass theorems): > *If $f(0) = 0, f_0(z) \neq 0$. Then in a smaller ball there is a $g(z,w) = g_w(z)$ such that $f = gh$ and $h(0) \neq 0$ is holomorphic. In other words, h is a unit in the local ring $\mathcal{O}_{\mathbb{C}^n,0}$ and $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z]$.*

Proof. We look at $f_w(z)$ as a family of holomorphic functions with zeros $\alpha_i(w)$ depending w. Put $g = \prod z - \alpha_i(w), h = f/g$. It turns out that the total count of such functions is independent of w. Now, if $f_w(a) = 0$ and $f_w(\xi) = \sum c_j(\xi - a)^j$ is its power series expansion, one sees that

$$\mathrm{Res}_{\xi=a}\xi^k \frac{f_w'(\xi)}{f_w(\xi)} = ma^k$$

where m is the first nonzero j, i.e. the multiplicity of the zero. The residue principle then says that

$$\sum \alpha_i(w)^k = \frac{1}{2\pi i}\int_{|\xi|=\epsilon} \xi^k \frac{f_w'(\xi)}{f_w(\xi)}$$

which is holomorphic in w, hence so is g. When $k = 0$ we see that d, the number of roots with multiplicity, is independent of w! □

Proposition 2.5 (Corollaries of WPT): *The local rings are UFDs and are Noetherian, using Weierstrass division. Irreducible Weierstrass polynomials are irreducible in these local rings. Moreover, if a germ f is irreducible in $\mathcal{O}_{\mathbb{C}^n,0}$, it remains irreducible in the nearby local rings $\mathcal{O}_{\mathbb{C}^n,z}$, perhaps outside a thin set Z. Similarly, two elements f,g are relatively prime in a local ring is a local property, perhaps outside of a thin set.*

Some other properties:

- Bijective holo maps are biholomorphisms!
- Codimension 1 analytic sets are defined by height 1 prime ideals, which by Krull's Hauptidealsatz are principal. Hence, hypersurfaces are locally defined by a single equation
- An analytic set is irreducible iff its ideal of functions vanishing on it $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is prime.

Proposition 2.6 (Order of vanishing): *Given an irreducible $g \in \mathcal{O}_{\mathbb{C}^n,0}$ and an f which vanishes on the zero set of g, then g divides f and in fact there is a well-defined order m such that $f = g^m f_0$ with f_0 not vanishing.*

Proof. Firstly, by the preparation theorem we may assume that $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z]$ and by Weierstrass division that $f = gh + r$. Now, r_w has degree less than g_w as polynomials of z and r_w vanishes whenever g_w vanishes. If, for a generic w, the roots of g_w have multiplicity 1, then for degree reasons we must have that $r_w = 0$. So we need to verify that for "most" w this is the case. Then we can conculde by the identity principle.

However, since g is irreducible and $\partial g/\partial z$ is of degree one less than g, by Gauss's lemma we can write

$$h_1 g + h_2 \frac{\partial g}{\partial z} = \gamma$$

where $\gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$. Thus, the set of w such that g_w has roots of multiplicity higher than one is contained in the zero set of γ which is a thin set! $\square$

2.2 Complex manifolds

A complex manifold is a smooth manifold M of even dimension such that its transition functions are holomorphic. Equivalently, their differentials commute with the standard almost complex structure on $\mathbb{R}^{2n}$.

Definition 2.7 (Holomorphic functions between complex manifolds): *$f : M \to N$ is holomorphic if one of the following equivalent statements is true:*

- *df commutes with J_M and J_N.*
- *df respects the splitting of the complexified tangent bundles.*

2.2.1 The almost complex structure on a complex manifold

To show the existence of an almost complex structure on X, we first define a canonical almost complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$, give it to the patches of X and then glue them together.

Take the canonical almost complex structure on $\mathbb{C}^n = \mathbb{R}^{2n}$ sending

$$(x_1, y_1, ..., x_n, y_n) \mapsto (-y_1, x_1, ..., -y_n, x_n)$$

Now define $I : T\mathbb{C}^n \to T\mathbb{C}^n$ to be the induced almost complex structure on the (smooth) tangent bundle of $\mathbb{C}^n$ by using the identifications $T_x\mathbb{C}^n \simeq \mathbb{C}^n$.

If $\phi : U \to \mathbb{C}^n, \psi : V \to \mathbb{C}^n$ are local trivializations of some U, V in a cover forming a complex structure on X, then the transition maps τ_{UV} are holomorphic. Hence $d\phi$ identifies T_U with $U \times \mathbb{C}^n$ and similarly with V, and does so in such a way that, at a given point x, the holomorphic map $d\tau_{UV}$ is $\mathbb{C}$–linear.

Now we can give almost complex structures $J^U \in End(T_U)$ and $J^V \in End(T_V)$ along the maps $d\phi$ and $d\psi$. More precisely, $J^U = d\phi^{-1} \circ I \circ d\phi, J^V = d\psi^{-1} \circ I \circ d\psi$. These glue together along $U \cap V$ for the following reason: on $U \cap V$, $d\phi = d\tau_{UV} \circ d\psi$ and hence $J^U = d\psi^{-1} \circ d\tau_{UV}{}^{-1} \circ I \circ d\tau_{UV} \circ d\psi = d\psi^{-1} \circ I \circ d\psi = J^V$ since by definition **??**, the holomorphicity of τ_{UV} means that $d\tau_{UV}$ commutes with I.

All in all, the separate I^U's glue together to form $J : T_X \to T_X$ with $J^2 = -1$, which is the almost complex structure on the tangent bundle of X.

2.2.2 The holomorphic tangent bundle

Define new operators as follows:

Definition 2.8 (Complex partials):

$$\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}),\ \frac{\partial}{\partial \overline{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})$$

This allows for a complex analogue of the usual Jacobian:

Definition 2.9 (Complex Jacobian): *The complex Jacobian is defined as*

$$J_{\mathbb{C}}(f)=(\frac{\partial f_i}{\partial z_j})_{i,j}$$

For holomorphic f, this is related to the real Jacobian as follows:

$$J_{\mathbb{R}}(f)=\begin{pmatrix} J_{\mathbb{C}}(f) & 0 \\ 0 & \overline{J_{\mathbb{C}}(f)} \end{pmatrix}$$

This shows that

$$\det J_{\mathbb{R}}(f)=|\det J_{\mathbb{C}}(f)|^2>0$$

and hence any complex manifold is orientable, since the structure group can be reduced from $GL(2n,\mathbb{R})$ to $GL_n(\mathbb{C})$
The off-diagonal entries correspond to $\overline{\partial f}$ and that is why theyre zero for f holomorphic.

Hence, on a complex manifold M with transition functions ϕ_{ij}, we can create a vector bundle whose cocycles are given by the complex Jacobians of the transition functions. This results in the holomorphic tangent bundle and similarly, by taking the conjugate we get the antiholomorphic tangent bundle. This splitting is reflected in the following way:

Definition 2.10 (Complexified tangent space): *$TM\otimes_{\mathbb{R}}\mathbb{C}$ is locally of the form $span_{\mathbb{C}}(\partial/\partial x,\partial/\partial y)$. The complex structure induces $J: TM\to TM$ sending $\partial/\partial x\mapsto\partial/\partial y, \partial/\partial y\mapsto -\partial/\partial x$. This has eigenvalues $\pm i$ on the complexified tangent spaces, which correspond to eigenspaces $T^{1,0}M$ and $T^{0,1}M$, the holomorphic and antiholomorphic tangent bundles. The former consists of vector fields killing antiholomorphic functions, and the latter to vector fields killing holomorphic functions, if we think of them as derivations.*

For example, if $Jv=-iv$ is an antiholomorphic vector field and f is holomorphic, then

$$Jv\cdot f=df(Jv)=-idf(v)=idf(v)\implies df(v)=0=v\cdot f$$

We used the complex linearity of df and the fact that $Jv=-iv$.

Definition 2.11 (Holomorphic vector field): *A section $\xi\in\Gamma(T^{1,0}M)$ is a holomorphic vector field if it preserves holomorphic functions. This is equivalent to its coefficients being holomorphic functions.*

Sections transform just like they do in differential geometry:

Formula (Transformation formula):

$$\frac{\partial}{\partial w_k} = \sum \frac{\partial z_i}{\partial w_k} \frac{\partial}{\partial z_i}$$

Definition 2.12 (Canonical decomposition):

$$\Lambda^r(T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=r} \Lambda^{p,q}(T^*M \otimes \mathbb{C})$$

$$\Lambda^{p,q}(T^*M \otimes \mathbb{C}) := \Lambda^p(T^*M)^{1,0} \wedge \Lambda^q(T^*M)^{0,1}$$

This also holds when we pass to sections, i.e. replace Λ by Ω. The statement of the Hodge decomposition theorem is that it also holds when we pass to cohomology, for certain complex manifolds (compact Kähler). The canonical line bundle is defined as the top exterior power of the holomorphic cotangent bundle:

$$K_M := \Lambda^{n,0} T^*M \otimes \mathbb{C} = \Lambda^n(T^*M)^{1,0}$$

This has transition functions given by the top exterior powers of the Jacobian, which is just the scalar multiplication by the determinant of the complex Jacobian.

Lemma 2.13 (Operators on complex manifolds): *Let M be a complex manifold, d the exterior derivative. Then the folllowing are true:*

- $d = \partial + \overline{\partial}$
- $d^2 = \partial^2 = \overline{\partial}^2 = 0$
- $\partial\overline{\partial} = -\overline{\partial}\partial$

Remark: pullbacks of holomorphic maps preserve the decomposition.

2.2.3 Poincare lemma and the analysis of Kähler metrics

The following lemma is fundamental in showing the exactness of the Dolbeault complex. We present the one-variable statement, which can be extended to the multivariable case as well.

Theorem 2.14 ($\overline{\partial}$-Poincare lemma): *Given $\alpha \in \mathcal{A}^{0,1}(U)$ (U is an open neighbourhood of a closed ball B) a holomorphic section of the form $f d\overline{z}$, then there exists a function given by a kernel such that $\overline{\partial} g = \alpha$.*

$$g := \frac{1}{2\pi i} \int_B \frac{f(w)}{w-z} dw \wedge d\overline{w}$$

More generally, given $\alpha \in \mathcal{A}^{p,q}(B)$ which is $\overline{\partial}$-closed, then $\exists \beta \in \mathcal{A}^{p,q-1}(B)$ such that $\overline{\partial}\beta = \alpha$.

Theorem 2.15 (Kähler forms osculate to the standard one): *Given g a metric an $\omega = g(J-,-)$ the associated fundamental form, then ω is closed if and only if for any point there exists local coordinates such that g osculates to order two to the standard metric.*

Proof. Consider $h = g - i\omega$ the Hermitian metric and its matrix representation $h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. Then

$$\omega = \frac{i}{2}\sum h_{ij} dz_i \wedge d\overline{z}_j$$

Translating the coordinates, assume $h_{ij}(0) = I$ and write

$$h_{ij} = \delta_{ij} + \sum a_{ijk} z_k + \sum a'_{ijk}\overline{z}_k + O(|z|^2)$$

Thus $a_{ijk} = \frac{\partial h_{ij}}{\partial z_k}(0)$ and similarly for a'_{ijk}. The closedness of ω implies that $a_{ijk} = a_{kji}, a'_{ijk} = a'_{ikj}$. Moreover, ω is real so $h_{ij} = \overline{h}_{ji}, a'_{ijk} = \overline{a}_{jik}$. We can now define new coordinates near the origin

$$w_j = z_j + \frac{1}{2}\sum a_{ijk} z_i z_k$$

since the Jacobian of the matrix of the w_i at 0 is the identity matrix. One then does a calculation to show that, up to terms of order at least two,

$$\frac{i}{2}\sum dw_j \wedge d\overline{w}_j = \omega$$

□

2.2.4 Examples of complex manifolds

Example (Complex projective space): The complex projective space $\mathbb{CP}^n$ is the set of lines in $\mathbb{C}^{n+1}$, in other words the quotient $\mathbb{C}^{n+1} \setminus 0/\mathbb{C}^*$. We can specify coordinates by saying that a line $l = [z_0 : \dots : z_n] = \langle (z_0, \dots, z_n) \rangle$, where at least one $z_i \neq 0$. This gives the canonical charts $U_i := \{z_i \neq 0\}$ and transition functions

$$\varphi_{ij}(w_1, \dots, w_n) = (\frac{w_1}{w_i}, \dots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \dots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_j}{w_i}, \dots, \frac{w_n}{w_i})$$

Another way to think about the transition functions is to identify $\varphi_i(U_i)$ as the set of complex numbers with $z_i = 1$. Then, $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ is the same as multiplying by z_i^{-1}. This description of the transition functions can be used to calculate the determinant bundle, i.e. the dual of the canonical bundle, and show it is equal to $\mathcal{O}(-n-1)$. More on this later.

Example (Grassmanian manifold): Similarly as above, one can see the set of k-planes $\mathrm{Gr}_k(\mathbb{C}^n)$ in complex n-space as a space topologized using the quotient topology on the frame bundle $\mathrm{Fr}_k(\mathbb{C}^n)$ consisting of k linearly independent vectors in $\mathbb{C}^n$. This is an open condition, so the frame bundle is topologized as a subspace of $\mathbb{C}^{kn}$ in its own right. Note that this is a principal bundle.

Let's say we have a k-tuple of vectors $w_1,...,w_n$ spanning W, which we can think of as a $k\times n$ matrix. By linear independence, this has some $k\times k$ submatrix which is nonsingular. These submatrices are indexed by the subsets $I \subset \{1,...,n\}$, and also correspond to a canonical set of k-planes, once we choose a basis $e_1,...,e_n$. We thus get an open cover $\cup U_I$, where U_I consists of the set of k-planes spanned by k vectors whose I-th minor A^I is nonsingular. There is a map

$$\begin{aligned} \phi_I : U_I &\to \mathrm{Mat}_{k,n-k} \\ \pi(A) &\mapsto (A^I)^{-1}(A^{I^c}) \end{aligned}$$

This is well-defined, since the quotient is modding out by multiplication by a $k\times k$ invertible matrix.

Example (Plücker embedding): Now, let's consider the Plucker embedding, where V is an n-dimensional complex vector space:

$$\begin{aligned} Gr_k(V) &\to \mathbb{P}(\bigwedge^k V) = Gr_1(\bigwedge^k V) \\ W \subset V &\mapsto \bigwedge^k W \subset \bigwedge^k V \end{aligned}$$

We have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fr}_k(V) \arrow[r] \arrow[d] & \mathrm{Fr}_1(\bigwedge^k V) \arrow[d] \\ \mathrm{Gr}_k(V) \arrow[r] & \mathrm{Gr}_1(\bigwedge^k V) \end{tikzcd}$$

What this amounts to in the local coordinates is the following: let's say $W = \langle w_1,...,w_k\rangle$ is described by the matrix A with rows w_i. Hence, in one chart given by the nonsingularity of A^I it is given by $(A^I)^{-1}A^{I^c}$. On the other hand, let's see what happens to the image $\bigwedge^k W$. This is a line spanned by the single vector $w_1 \wedge ... \wedge w_k$. In standard coordinates $\{e_I = e_{i_1} \wedge ... \wedge e_{i_k}\}$, this can be described as

$$w_1 \wedge ... \wedge w_k = \sum |A^I| e_I$$

This is a $1\times\binom{n}{k}$ matrix, with one entry for each I. In the chart given by the nonvanishing of $|A^I|$, i.e. the nonsingularity of the 1×1 matrix $(|A^I|)$, we have that this is sent to some number in $\mathbb{C}^{1.(\binom{n}{k}-1)}$ given by

$$|A^I|^{-1}(|A^J|, J \neq I)$$

All in all, in these charts we get a map

Example (Continued):

$$\mathbb{C}^{k(n-k)} \to \mathbb{C}^{\binom{n}{k}-1}$$
$$(A^I)^{-1}A^{I^c} \mapsto |A^I|^{-1}(|A^J|, J \neq I)$$

In practice, this takes a $k \times (n-k)$ matrix C, then forms all of the $k \times k$ matrices using at least one column from C, plus a bunch of simple entries coming from the identity $k \times k$ matrix, then evaluates all of their determinants. This is clearly holomorphic, since it is all defined by algebraic equations.

All in all, this means that the map sending a frame of W to the line spanned by the determinants of the minors lifts the Plucker embedding:

$$Fr_k(V) \mapsto Gr_1(\bigwedge^k V) \simeq \mathbb{CP}^{\binom{n}{k}-1}$$
$$A = \begin{pmatrix} w_1 \\ w_2 \\ \dots \end{pmatrix} \mapsto [\dots : |A^I| : \dots]$$

At least one of the A^I is going to be nonsingular, since the w_i are linearly independent, so this is well-defined. Moreover, it descends to the Grassmanian, where we have quotiented by the $GL_k(\mathbb{C})$ action since the maps is equivariant: multipliying by Q turns A^I into QA^I, so everything is multiplied by $|Q|$.

Recall that after choosing a basis e_i, there is a canonical set of k-planes spanned by size k subsets I of the basis. Each such plane is given by the span of $\{e_i\}_{i\in I}$ and has an $k(n-k)$ parameter neighbourhood given by altering the entries like in the following example:

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \end{pmatrix}, s, t \in \mathbb{C}$$

We have $\mathrm{Gr}_2(\mathbb{C}^3)$ and are looking in a neighbourhood of the 2plane spanned by e_1, e_2. This has then a neighbourhood biholomorhpic to $\mathbb{C}^{2.(3-2)}$, given by the parameters s, t, which are really a set of k $n-k$-vectors. The $(A^I)^{-1}$ is there to signify a change of basis/ mod out by the action of GL_k. In general, any k-plane is going to have some neighbourhood of this type by choosing coordinates properly. Now, one can see that a path of k-planes near this one is given by something like

$$\begin{pmatrix} 1 & 0 & \gamma_1(t) \\ 0 & 1 & \gamma_2(t) \end{pmatrix}, t \in I$$

The effect of the map in the Plucker embedding is going to send it to a bunch of determinants which in this case are equal to $(\gamma_2(t), -\gamma_1(t))$. When we take derivatives, we see that this vanishes precisely when $\dot{\gamma_i} = 0$ so the map is injective on tangent spaces. One can do the same thing to arbitrary cases like this, and see that all of the vectors $\gamma_i(t)$ have vanishing derivative. In a coordinate free way, one has to show that the map on tangent spaces is given by

$$\mathrm{Hom}(W, V/W) = \mathrm{Hom}(W, W^\perp) \to \mathrm{Hom}(\bigwedge^k W, (\bigwedge^k W)^\perp) = \mathrm{Hom}(\bigwedge^k W, (\bigwedge^k V)/\bigwedge^k W))$$

2.3 Almost complex manifolds and integrability

Integrability etc.

2.4 Holomorphic vector bundles: Normal bundle sequence, adjunction formula, Picard group and sheaves

2.4.1 Dolbeault cohomology and the Poincare lemma

Definition 2.16 (Dolbeault cohomology): *$H^{p,q}(M)$ is defined as the cohomology of this complex (fixed p, varying q): >*

$$\dots \xrightarrow{\overline{\partial}} \Omega^{p,q-1} \xrightarrow{\overline{\partial}} \Omega^{p,q} \xrightarrow{\overline{\partial}} \dots$$

2.5 Divisors and line bundles

A hypersurface $Y \subset X$ is a codimension one analytic subvariety, which means that it is locally given by the vanishing of a single holomorphic function. It turns out that every hypersurface can be considered as the vanishing locus of a global function, which is however not a map to $\mathbb{C}$, but a section of a line bundle!

2.5.1 Weil divisors

Definition 2.17 (Divisors):
A divisor is a formal linear combination

$$D = \sum a_i [Y_i]$$

of irreducible hypersurfaces, which is locally finite. These form the divisor group $\mathrm{Div}(X)$*. D is called effective if $a_i \geq 0$.*

Definition 2.18 (Order of a meromorphic function along a hypersurface): *Let f be a meromorphic function defined near $y \in Y$. Here Y is defined as the zero locus of some irreducible $g \in \mathcal{O}_{X,y}$ and the different choices of g differ only by a unit. Hence, by 2.6 we see that $f = g^m h$ and we define m to be $\mathrm{ord}_{Y,y}(f)$. This is clearly additive. By the corollary of WPT, we see that this value is also locally constant when Y is irreducible, since then Y_{reg} is connected, hence it becomes independent of y. This is fundamentally we restrict to irreducible hypersurfaces in the definition of divisors and allows for the next definition.*

Definition 2.19 (Principal divisors): *For a meromorphic function we put*

$$\mathrm{div}(f) := \sum \mathrm{ord}_Y(f)[Y]$$

where the sum ranges over all irreducible hypersurfaces Y. It decomposes into a positive part (zero divisor) and negative part (pole divisor) and hence is effective precisely when f is holomorphic. Moreover, it is finite, as the vanishing locus of f contains only finitely many irreducible hypersurfaces why?.

Definition 2.20 (Line bundle associated to a Weil divisor): *Let $D = \sum a_i[Y_i]$ be a Weil divisor. We define*

$$\mathcal{O}(D)(U) := \{f \in K(X)^\times | (\mathrm{div}(f) + D)|_U \geq 0$$

This consists of the meromorphic functions constrained by the divisor D, where positive coefficients of D allow for poles of that order and negative coefficients require a zero of at least that order.

We have just been talking about what in algebraic geometry is called a Weil divisor. We now tie it up with the notion of a Cartier divisor:

2.5.2 Cartier divisors

Definition 2.21 (Cartier divisors): *A Cartier divisor is a section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. It can be thought of as a system $\{U_i, f_i\}$ where each f_i is a nontrivial meromorphic function on U_i and such that f_i/f_j is a holomorphic unit, i.e. has no zeros.*

Proposition 2.22 (Cartier = Divisor in the complex setting):

$$H^0(\mathcal{K}_X^*/\mathcal{O}_X^*) \simeq \mathrm{Div}(X)$$

Proof. Given a Cartier divisor $\{U_i, f_i\}$ we associate to it the Weil divisor glued from the $\operatorname{div} f_i$. These glue for the following reason: firstly, f_i/f_j is a holomorphic unit, since both functions define Y and must vanish to the same order on it, i.e. differ by units in the local rings, and are invertible outside Y. Hence, we must have that $\operatorname{div}(f_i) = \operatorname{div}(f_j)$ on $U_i \cap U_j$. This gives the homomorphism from left to right. We now need to show it is an isomorphism.

Consider $D = \sum a_i[Y_i]$ where Y_i is given on U_j locally by g_{ij}, unique up to a unit. We can then form $f_j = \prod g_{ij}^{a_i}$ and this forms a Cartier divisor, since g_{ij} and g_{ik} produce the same hypersurface on $U_j \cap U_k$ and hence differ by a holomorphic unit, which is precisely what a Cartier divisor does. □

In the following pages, we will often use this identification between Weil and Cartier divisors. However, for clarity in the situations in algebraic geometry when the two do not coincide, we will keep track of the notation and write Cartier(X) for the Cartier divisors and Div(X) for the Weil divisors. We will write $\mathcal{D}$ for a Cartier divisor corresponding to the Weil divisor D.

Definition 2.23 (Line bundle associated to a Cartier divisor): *Given $\mathcal{D} = \{U_i, f_i\}$ a system defining a Cartier divisor, we define the line bundle $\mathcal{O}_X(\mathcal{D})$ using the cocycle $f_i f_j^{-1}$. Hence, $\mathcal{O}(\mathcal{D})(U_i) = \frac{1}{f_i}\mathcal{O}_X(U_i)$. This produces a homomorphism* $\mathrm{Div}(X) \simeq \mathrm{Cartier}(X) \to \mathrm{Pic}(X)$. *We will show that the image of this homomorphism consists of line bundles admitting a global section.*

2.5.3 Relationship with the Picard group

Recall that Pic(X) is the group of line bundles on X with the tensor product operation and is isomorphic to $H^1(X, \mathcal{O}_X^*)$. We have the exponential sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \to 0$$

Definition 2.24 (First Chern class): *We define $c_1(L)$ to be the image of L in $H^2(X;\mathbb{C})$ in the exponential sheaf sequence. We will embark to show that $c_1(\mathcal{O}(Y))$ is Poincare dual to the hypersurface Y.*

We will show later that for projective space the Chern homomorphism is an isomorphism, i.e. line bundles over $\mathbb{CP}^n$ are classified by their Chern number, which is the integer d such that $L \simeq \mathcal{O}(d)$.

We have seen that there is a natural map $\mathrm{Cartier}(X) \to \mathrm{Pic}(X)$. We will now see that it is natural, at least for dominant maps. Given a dominant map (i.e. with dense image) $f : X \to Y$ and a Cartier divisor $\mathcal{D} = \{f_i, U_i\}$ the pullback divisor is given by $f^*\mathcal{D} = \{f_i \circ f, f^{-1}(U_i)\}$. For a Weil divisor D, we need to see what happens to hypersurfaces $Z \subset Y$. The pullback $f^*[Z] = \sum n_j[H_j]$ where n_j are such that $g \circ f = \prod g_j^{n_j}$, with g a local defining function of Z. Under the isomorphism, these two definitions agree. Moreover, we have a natural square

$$\begin{tikzcd}
\mathrm{Div}(Y) \arrow[d, "f^*"'] \arrow[dr, "\simeq"] \arrow[rr, dashed, bend left] & & \mathrm{Pic}(Y) \arrow[d, "f^*"] \\
\mathrm{Div}(X) \arrow[dr, "\simeq"'] \arrow[rr, dashed, bend left] & \mathrm{Cartier}(Y) \arrow[ur, "\mathcal{O}(-)"] \arrow[d, "f^*"'] & \mathrm{Pic}(X) \\
& \mathrm{Cartier}(X) \arrow[ur, "\mathcal{O}(-)"'] &
\end{tikzcd}$$

Lemma 2.25 (Canonical homomorphism factors through principal divisors): *A Weil divisor D is principal if and only its associated Cartier divisor $\mathcal{D}$ has $\mathcal{O}(\mathcal{D}) \simeq \mathcal{O}_X$,*

Proof. Suppose the associated Cartier divisor is given by $\{U_i, f_i\}$ and its line bundle with cocycles $f_i f_j^{-1}$ is trivial. Then, these cocycles are cohomologous to a coboundary, i.e. $f_i/f_j = g_i/g_j$ for some holomorphic g_i. But then the $f_i g_i^{-1}$ glue to a global meromorphic f whose principal divisor is D.

Conversely, if the divisor is principal, then its Cartier divisor is just given by the identity cocycles. □

Hence, we get an induced injective map $Div(X)/\mathrm{PrinDiv}(X) \to \mathrm{Pic}(X)$, whose image we want to show consists of line bundles admitting a section. Note that the map defined is the boundary map in the long exact sequence of

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$$

2.5.4 Sections of line bundles

Suppose we have a nontrivial meromorphic section $s : X \to L$. If L is trivialized using the maps $\psi_i : L|_{U_i} \simeq \mathcal{O}|_{U_i}$, then its cocycle is given by $\psi_i \circ \psi_j^{-1}$. Using these trivializations, s becomes an ordinary function $\psi_i(s) = f_i$ on U_i. Then we have that $f_i f_j^{-1} = \psi_{ij}$, which is a holomorphic unit and hence the system $\{U_i, f_i\}$ defines a Cartier divisor. We denote this divisor by $(s) \in \mathrm{Cartier}(X)$.

$$\begin{tikzcd}
(U_i \cap U_j) \times \mathbb{C} \arrow[rr, bend left, "(id,\psi_{ij})"] & L|_{U_i \cap U_j} \arrow[l, "\psi_j"'] \arrow[r, "\psi_i"'] \arrow[d, shift left] & (U_i \cap U_j) \times \mathbb{C} \\
& U_i \cap U_j \arrow[ul, "(id,f_j)"] \arrow[ur, "(id,f_i)"'] \arrow[u, bend left, "s"] &
\end{tikzcd}$$

In terms of Weil divisors, this is also

$$\sum \mathrm{ord}_Y(s)[Y]$$

where $\mathrm{ord}_Y(s) := \mathrm{ord}_Y(\psi_i(s))$ does not depend on the trivialization. Moreover, this is independent of whether we scale s by a nonzero complex number.

Proposition 2.26 (Divisors of sections): *We have that $\mathcal{O}((s)) \simeq L$. Moreover, any effective divisor occurs as (s), for some section of its associated line bundle. Two sections s_1, s_2 of line bundles L_1, L_2 define linearly equivalent diviors if and only if $L_1 \simeq L_2$.*

Proof. The first part follows by definition: given s defining a Cartier divisor $\{U_i, f_i = \psi_i(s|_{U_i})\}$ we have just shown that $f_i/f_j = \psi_{ij}$ which are the cocycles for L and the associated line bundle $\mathcal{O}(s)$. If D is associated to $\mathcal{D} = \{U_i, f_i\}$ then since D is effective, the f_i are holomorphic and define a section s of $\mathcal{O}(\mathcal{D})$ as the transition functions of this line bundle are given by f_i/f_j, and then $(s) = D$. The last part follows since $\mathcal{O}(s) \simeq L$. □

All in all, we obtain that the image of the map $\mathrm{Cartier}(X) \to \mathrm{Pic}(X)$ is given by the line bundles admitting a nonzero global section.

Example (Hyperplane in projective space): The hyperplane $H_k = \{z_k = 0\}$ is defined in the open cover U_i by the functions $g_i = z_k/z_i$. Therefore, the line bundle associated to it will have transition functions z_j/z_i and hence is isomorphic to $\mathcal{O}(1)$.

Now we move on to show that sections of line bundles allow for maps to projective space, at least away from the common zero locus.

Proposition 2.27 (Sections and maps to projective space): *Given $s_0, \ldots, s_n : X \to L$ sections of a line bundle L with common zero locus $B = Z(s_0) \cap \ldots \cap Z(s_n)$ there is an induced map*

$$\varphi_L : X \setminus B \to \mathbb{CP}^n, x \mapsto [s_0(x) : \ldots : s_n(x)]$$

What this means is that we consider the images of the s_i in a particular trivialization, and different trivializations differ by a scalar multiple, hence give the same element in projective space. Moreover, $\varphi_L^* \mathcal{O}(1) \simeq L|_{X \setminus B}$

To see the last claim, consider z_i as a section of $\mathcal{O}(1)$ whose associated divisor is given by the hypersurface $H = \{z_i = 0\}$. Then $f^* H_i = (s_i)$ and hence $\mathcal{O}((s_i)) = \mathcal{O}(f^* H_i) = f^* \mathcal{O}(1)$ where φ_L is defined.

Example (Veronese embedding): When we consider $H^0(\mathbb{CP}^n, \mathcal{O}(d))$, we will later show that this consists of homogenous polynomials of degree d in $n+1$ variables. If we consider all monomials spanning this, what we get is the Veronese embedding!

Example (Segre embedding): The line bundle $\pi_1^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(1)$ on $\mathbb{CP}^n \times \mathbb{CP}^m$ has linearly independent sections $z_i z_j'$, defining a map

$$\mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{(n+1)(m+1)-1}, ([z_0 : \ldots : z_n], [z_0', \ldots, z_m']) \mapsto [z_0 z_0' : \ldots : z_n z_m']$$

2.5.5 Adjunction formulae

Recall that for a subvariety Y we have the normal bundle sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0$$

When Y is a hypersurface, the normal bundle is a line bundle. When Y is a complete intersection of hypersurfaces Y_i, the normal bundle is the sum of the normal line bundles of the Y_i.

By dualizing and taking top exterior powers, we see that $\det \mathcal{N}^* \otimes K_Y \simeq K_X|_Y$.

Proposition 2.28 (Adjunction formulas): *When Y is a hypersurface*

$$\mathcal{O}(Y)|_Y \simeq \mathcal{N}_{Y/X}, K_Y \simeq (K_X \otimes \mathcal{O}(Y))|_Y$$

Proof. We calculate cocycles.

Let X have transition functions φ_{ij} which send Y to the hyperplanes $z_n = 0$. Then the holomorphic tangent bundle has cocycle given by the derivatives, i.e. the complex Jacobians $J(\varphi_{ij})$, composed with the trivialization φ_j. We will show that these Jacobians, restricted to Y, look like

$$\begin{pmatrix} J(\varphi_{ij}|_Y) & * \\ 0 & \frac{\partial \varphi_{ij}^n}{\partial z_n} \end{pmatrix}$$

Hence, the normal bundle is given by the cocycle $\frac{\partial \varphi_{ij}^n}{\partial z_n} \circ \varphi_j|_Y$. The reason for this is that the transition functions φ_{ij} send $z_n = 0$ to 0, i.e. they preserve the hyperplanes that Y corresponds to: $\varphi_{ij}^n(z_1, ..., z_{n-1}, 0) = 0$. Hence, there is some holomorphic function h such that $\varphi_{ij}^n = z_n h$ and thus

$$\frac{\partial \varphi_{ij}^n}{\partial z_k}(z_1, ..., z_{n-1}, 0) = 0$$

for $k \neq n$ and is equal to h for $k = n$. (Note that putting $z_n = 0$ is the same as restricting to Y).

On the other hand, Y is defined by the functions φ_i^n. Let $y \in Y$ such that $\varphi_j(y) = (z_1, ..., z_{n-1}, 0)$, Then the line bundle associated to Y has cocycles

$$\frac{\varphi_i^n}{\varphi_j^n}(y) = (\frac{\varphi_{ij}^n}{z_n} \circ \varphi_j)(y) = h(z_1, ..., z_{n-1}, 0) = (\frac{\partial \varphi_{ij}^n}{\partial z_n} \circ \varphi_j)(y)$$

The second part follows from the fact that $K_Y \simeq K_X|Y \otimes \det \mathcal{N}$. □

Remark : We have just seen that $\mathcal{O}(Y)|_Y$ is the normal bundle to Y. Outside Y, the bundle is trivial, as the defining cocycle f_i/f_j is actually a coboundary, since the defining functions are nonvanishing, i.e. holomorphic units, outside of Y! So we can characterize $\mathcal{O}(Y)$ as a line bundle which is trivial outside Y and the normal bundle to Y over Y.

2.5.6 The case of complex curves

See the notes on abelian varieties for a recap of this.

2.5.7 Line bundles on projective space

We now describe some properties of line bundles on $\mathbb{CP}^n$.

Firstly, note that any linear homogenous polynomial $\sum a_i z_i$ can be thought of as a linear map $\mathbb{C}^{n+1} \to \mathbb{C}$. We will describe an isomorphism $H^0(\mathbb{P}(V), \mathcal{O}(1)) \simeq V^*$.

More generally, suppose we are given a homogenous polynomial p in $n+1$ variables of degree d. This defines a linear map $(\mathbb{C}^{n+1})^{\otimes k} \to \mathbb{C}$ by symmetrization/polarization. For example, $z_0 z_1$ is a homogenous degree 2 polynomial and hence defines a linear map by $\frac{1}{2}(z_0 \otimes z_1 + z_1 \otimes z_0) : v \otimes w \mapsto \frac{1}{2}(v_0 w_1 + v_1 w_0)$.

More generally, the linear map is defined by

$$P(v_0, ..., v_n) = \frac{1}{d!} \frac{\partial^d}{\partial t_1 ... \partial t_d}|_{t=0} p(t_1 v_1 + ... + t_d v_d)$$

Hence, any such homogenous polynomial defines a holomorphic map $\mathbb{CP}^n \times (\mathbb{C}^{n+1})^{\otimes k} \to \mathbb{C}$ which when restricted to $(\mathcal{O}(-1))^{\otimes k}$ produces a section of $\mathcal{O}(k)$. We now show that these are all the sections.

Proposition 2.29 (Sections of line bundles on projective space):

$$H^0(\mathbb{CP}^n, \mathcal{O}(k)) \simeq \mathbb{C}[z_0, \ldots, z_n]_k$$

Proof. Let $t, s \in H^0(\mathbb{CP}^n, \mathcal{O}(k))$ be two sections such that s is induced by a homogenous polynomial. Then t/s defines a meromorphic function on $\mathbb{CP}^n$, due to the general fact that two sections t, s of line bundles L, L' define a meromorphic section t/s of $L \otimes (L')^*$. Composing with the quotient map we obtain a meromorphic map F on $\mathbb{C}^{n+1} \setminus 0$. Now, think again of s as a homogenous polynomial and hence $G = sF$ is a holomorphic function on $\mathbb{C}^{n+1} \setminus 0$ which by Hartogs' theorem 2.2 can be extended to a holomorphic function on $\mathbb{C}^{n+1}$. By definition of F as a composition with the quotient map, we have that $F(\lambda z) = F(z)$ whereas $s(\lambda z) = \lambda^k s(z)$ and hence G has $G(\lambda z) = \lambda^k G(z)$. By examining the power series of G this shows that G is homogenous of degree k, and the section induced by G is precisely t. ☐

We thus see that the polynomial ring can be recovered as

$$\bigoplus H^0(\mathbb{CP}^n, \mathcal{O}(k)) \simeq \mathbb{C}[z_0, \ldots, z_n]$$

Proposition 2.30 (Canonical bundle of projective space): *The canonical bundle of projective space is*

$$K_{\mathbb{CP}^n} \simeq \mathcal{O}(-n-1)$$

Proof. Again, we do a cocycle computation. The tangent bundle is given by the Jacobian of the transition functions composed with one of them, which for projective space look like 2.2.4. Thus, we just need to take the determinant of this and compose with φ_j and then invert it to get the cocycle of the canonical bundle. However, note that this Jacobian looks like $1/w_i$ along the diagonal except for one $-1/w_i^2$ which occurs in a column where the other entries are w_j/w_i^2. All of the rest are 0 so the determinant is $\pm w_i^{-n-1}$ which when composed with φ_j gives $\pm(z_i/z_j 0^{-n-1}$. The sign doesn't matter as it is a coboundary and lo and behold we get the cocycle for $\mathcal{O}(-1)^{\otimes -n-1} \simeq \mathcal{O}(n+1)$, which is also the cocycle for $\det \mathcal{T}_{\mathbb{CP}^n}$. Dualizing gives the result. ☐

Example (Canonical bundle of complete intersection in projective space): Firstly, suppose H is a hypersurface defined by an equation of degree d in $\mathbb{CP}^n$. Then $\mathcal{O}(H) \simeq \mathcal{O}(d)$ (see 2.5.4) and by the adjunction formula we get that

$$K_H \simeq (K_{\mathbb{CP}^n} \otimes \mathcal{O}(H))|_H \simeq \mathcal{O}(d-n-1)|_H$$

More generally, given a compete intersection $Y = \cap H_i$ with each H_i a hypersurface defined by an equation of degree d_i, then Y has a normal bundle which is the sum of the normal bundles of the H_i. Hence, from the conormal sequence we get $K_Y \simeq (K_{\mathbb{CP}^n})_Y \otimes \det \mathcal{N}_{Y/\mathbb{CP}^n}$. But now since $\mathcal{N}_{Y/\mathbb{CP}^n} = \oplus \mathcal{N}_{H_i/\mathbb{CP}^n}$ we get

$$\det(\mathcal{N}_{Y/\mathbb{CP}^n}) \simeq \bigotimes \mathcal{N}_{H_i/\mathbb{CP}^n} \simeq \bigotimes \mathcal{O}(d_i)|_Y \simeq \mathcal{O}(\sum d_i)|_Y$$

All in all, $K_Y \simeq \mathcal{O}(\sum d_i - n - 1)_Y$.

Proposition 2.31 (Euler sequence): *We have an exact sequence*

$$0 \to \Omega^1 \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

Proof. The proof follows the same outline as 3.76. We have an inclusion $\mathcal{O}(-1) \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ which when twisted with $\mathcal{O}(1)$ and dualized gives a map $\mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O}$. We need to check that the kernel of this is Ω^1. □

Remark (Remark): Note that dualizing the Euler sequence gives us another way to interpret the splitting

$$T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \simeq (\gamma^\vee)^{\oplus n+1}$$

that was done in K-theory and also Differential Geometry (see 1.5.2 and also the subsection on calculations of tangent bundles in K-theory).

If we assume some results from Dolbeault cohomology and the Kähler form on projective space (all of this is included later), we can even prove:

Theorem 2.32 (Picard group of projective space):

$$\operatorname{Pic}(\mathbb{CP}^n) \simeq \mathbb{Z}\omega = H^2(\mathbb{CP}^n; \mathbb{Z})$$

and moreover the isomorphism is given by the first Chern class $c_1(\mathcal{O}(d)) = d\omega$, where ω is the Fubini-Study form.

Proof. We have holomorphic local trivializations on L given by nowhere vanishing functions, i.e. L looks like $(x, \langle e^{f_\alpha}(x)\rangle)$ on U_α. In other words, this is a line bundle with cocycles given by $\varphi_{\beta\alpha} =$

e^{f_α}/e^{f_β}, as in a Cartier divisor, so we have that $f_\alpha - f_\beta$ is holomorphic, i.e.

$$\overline{\partial} f_\alpha = \overline{\partial} f_\beta$$

on the overlaps. Therefore, the $\overline{\partial}$-closed $(0,1)$ forms $\overline{\partial} f_\alpha$ glue to a global $\overline{\partial}$-closed $(0,1)$ form $\mathfrak{f}$, but since $H^{0,1}(\mathbb{CP}^n) = 0$ (by the Hodge decomposition, or LES associated to the Euler sequence), we must have that it is exact, i.e.

$$\mathfrak{f} = \overline{\partial} f$$

for some smooth function f on $\mathbb{CP}^n$. Therefore, $\overline{\partial} f - \overline{\partial} f_\alpha = 0$ on the patches U_α, hence $f - f_\alpha$ is holomorphic on U_α and this provides a coboundary to turn $\varphi_{\beta\alpha}$ into the trivial cocycle: putting $\Psi_\alpha = e^f/e^{f_\alpha}$ we get that

$$\varphi_{\beta\alpha} = \frac{\Psi_\beta}{\Psi_\alpha}$$

This then shows that any smoothly trivial line bundle is holomorphically trivial. Moreover, any topological line bundle is approximated by a smooth line bundle. This shows that the groups of holo, smooth and topological complex line bundles are all isomorphic!

$$L' \simeq_{holo} L \iff L' \otimes L^* \simeq_{holo} 1 \iff L' \otimes L^* \simeq_{top} 1 \iff L \simeq_{top} L'$$

But now, complex line on any space X bundles are classified by $H^2(X;\mathbb{Z})$ since $\mathbb{CP}^\infty$ is both BU and $K(\mathbb{Z},2)$:

$$\{\text{complex line bundles up to iso}\} \simeq [X, \mathbb{CP}^\infty] \simeq H^2(X;\mathbb{Z})$$

Hence, in the case of $\mathbb{CP}^n$, we have that the line bundles are isomorphic to $H^2(X;\mathbb{Z}) \simeq \omega\mathbb{Z}$, with ω the Fubini-Study form, and the isomorphism is given by the Chern class!

□

Remark (Exponential sequence): A much quicker proof is to use the exponential sequence and the fact that $h^{0,1} = h^{0,2} = 0$ and hence the Chern class homomorphism is an isomorphism.

2.6 Blowups

Let $\Delta \subset \mathbb{C}^n$ be a polydisk about 0. We define:

$$\tilde{\Delta} := \{(z, w) \in \Delta \times \mathbb{CP}^{n-1} | z_i w_j = z_j w_i\}$$

In other words, $z \in \langle w \rangle$. This comes equipped with a projection $\sigma : \tilde{\Delta} \to \Delta$, $(z, w) \to z$ and this is the blowup of Δ at 0. It is a homeomorphism away from 0, i.e.

$$\sigma : \tilde{\Delta} \setminus \sigma^{-1}(0) \simeq \Delta \setminus \{0\}$$

On the other hand, the preimage of 0 is the whole of $\mathbb{CP}^{n-1}$, which is called the **exceptional divisor**. This recovers the tautological line bundle when $\Delta = \mathbb{C}^n$.

$\tilde{\Delta}$ is a complex manifold, trivialized over the open cover $\Delta \times U_j \cap \tilde{\Delta}$ via $(z, w) \mapsto (\phi_j(w), z_i)$, where $\phi_j : U_i \simeq \mathbb{CP}^{n-1}$ is the standard trivialization of complex projective space.

To generalize to an arbitrary complex manifold X, consider $x \in U \subset X, \varphi : U \simeq \Delta$ a chart where x corresponds to 0. Then we can define

$$\tilde{X} := X \setminus \{x\} \cup_{\varphi^{-1} \circ \sigma} \tilde{\Delta}$$

Away from x, $\tilde{\Delta}$ is glued to X via the identity, i.e. does not change anything to X. However, we have replaced x by a copy of $\mathbb{CP}^{n-1}$. This defines the blowup $\sigma_x : \tilde{X} \to X$, which is independent of the choice of chart U. This is by the following lemma:

Lemma 2.33 (Naturality of blowup): *Suppose Δ, Δ' is the same polydisk with different coordinates, i.e. we have a biholomorphism $f : \Delta \to \Delta'$. Then this lifts naturally to a biholomorphism $F : \tilde{\Delta} \to \tilde{\Delta}'$ as in the diagram*

$$\begin{tikzcd}
\tilde{\Delta} \arrow[r, "F"] \arrow[d, "\sigma"'] & \tilde{\Delta}' \arrow[d, "\sigma'"] \\
\Delta \arrow[r, "f"] & \Delta'
\end{tikzcd}$$

Proof. We define $F(z, w) = (z', w')$ via $z' = f(z)$ and $w'_j = \sum \frac{\partial f_j}{\partial z_i}(0) w_i$. First assume f is linear given by a matrix A, in which case everything commutes as

$$z'_i w'_j = \sum A_i^k z_k \sum A_j^l w_l = \sum A_i^k A_j^l z_k w_l = \sum A_i^k A_j^l z_l w_k = \sum A_i^k w_k \sum A_j^l z_l = w'_i z'_j$$

Now for general f can compose with a linear map so that $df(0) = I$. This then allows us to show that $dF_p = I$ so by the inverse function theorem it is a biholomorphism. □

Let's now define blowups more generally for an arbitrary submanifold $Y \subset X$. They look locally like the inclusion $\mathbb{C}^m \subset \mathbb{C}^n$ so we better deal with that case first.

Remark: Important! Here, I am following Huybrechts, who reverses the roles of $\mathbb{CP}$ and $\mathbb{C}$, i.e. here we have the line w first and then the element z, whereas in lectures we did it the opposite way. Both are fine, just be wary of the swapping.

Definition 2.34 (Blowup along a linear subspace):

$$\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) := \{(w,z) \in \mathbb{CP}^{n-m-1} \times \mathbb{C}^n | z_i w_j = z_j w_i, i, j = m+1, \dots, n\}$$

Here, $\mathbb{C}^m$ is given by the vanishing of $z_{m+1} = \dots = z_n = 0$ and $[w_{m+1} : \dots : w_m]$ are the coordinates on projective space. In other words, this is the incidence variety of pairs of a vector and a line such that z is in $\{\mathbb{C}^m, \langle w \rangle\}$. This comes equipped with two projections, and the map $\sigma : \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$ is a biholomorphism away from $\mathbb{C}^m$ and is $\mathbb{P}(\mathcal{N}_{\mathbb{C}^m/\mathbb{C}^n})$ over it, where the normal bundle is trivial $\mathbb{C}^m \times \mathbb{C}^{n-m}$. I.e. over $\mathbb{C}^m$ the last $n-m$ coordinates of z are all 0 so we get all lines in $\mathbb{C}^{n-m}$ which is precisely the definition of the projective bundle of the trivial bundle.

More generally, let $X = \bigcup U_i, \varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n, \varphi_i(U_i \cap Y) = \varphi_i(U_i) \cap \mathbb{C}^m$. We can restrict the blowup of $\mathbb{C}^n$ along $\mathbb{C}^m$ to the subspaces $\varphi_i(U_i)$, i.e. get a pullback of blowups

$$\begin{array}{ccc} Z_i & \longrightarrow & \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \\ {\scriptstyle \sigma_i}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ \varphi_i(U_i) & \longrightarrow & \mathbb{C}^n \end{array}$$

These glue together, via a similar naturality argument as before. The idea is that if $U, V \subset \mathbb{C}^n$ are the images of two charts, then we have a bihilomorphism preserving Y, in other words a map $\phi : U \simeq V, \phi(U \cap \mathbb{C}^m) = V \cap \mathbb{C}^m$. This means that when the last $n-m$ coordinates are zero, the result is zero, and hence $\phi^k = \sum_{j=m+1}^n z_j \phi_{k,j}$ for $k > m$. This allows us to define a map between the blowups, i.e. the incidence varieties as follows:

$$\hat{\phi} : \sigma^{-1} U \to \sigma^{-1} V$$
$$\hat{\phi}(x,z) = ((\phi_{k,j}) \cdot x, \phi(z))$$

This works, as if $z \in \langle \mathbb{C}^m, \langle x \rangle \rangle$, then $z = (z_1, \dots, z_m, \lambda x_0, \dots, \lambda x_{n-m-1})$. Thus, $\phi^k(z) = \sum_j (\lambda x_j) \phi_{k,j}(z)$ for $k > m$ and hence $\phi(z) \in \langle \mathbb{C}^m, \langle ((\phi_{k,j})(z)) \cdot x \rangle \rangle$, where $\cdot$ means matrix product.

In each chart we are blowing up the part that corresponds to $\mathbb{C}^m$ and we get gluing maps ϕ, which lift to maps between the blowups. Outside of the parts which correspond to the submanifold Y, illustrated as the dotted set, these are uninteresting. However, over Y, $\phi_{k,j}$ becomes the cocycle for the normal bundle (compare with 2.28) and hence $\hat{\phi}$ the cocycle for $\mathbb{P}(\mathcal{N}_{Y/X})$.

In other words, $\mathrm{Bl}\, U \to U$ is a biholomorphism outside of the bit corresponding to Y, and is a projectivized normal bundle over Y, whence the matrix $\phi_{k,j}$ describes how the fibres transfor between the trivializations for U and V. We thus get:

Proposition 2.35 (Blowup along a submanifold): *Let Y be a complex submanifold of X. Then there exists a complex manifold $\mathrm{Bl}_Y(X) \xrightarrow{\sigma} X$ such that $\sigma : \mathrm{Bl}_Y(X) \setminus \sigma^{-1}(Y) \simeq X \setminus Y$ and $\sigma : \sigma^{-1} Y \to Y$ is equal to $\mathbb{P}(\mathcal{N}_{Y/X}) \to Y$. We call $\sigma^{-1}(Y)$ the exceptional divisor of the blowup.*

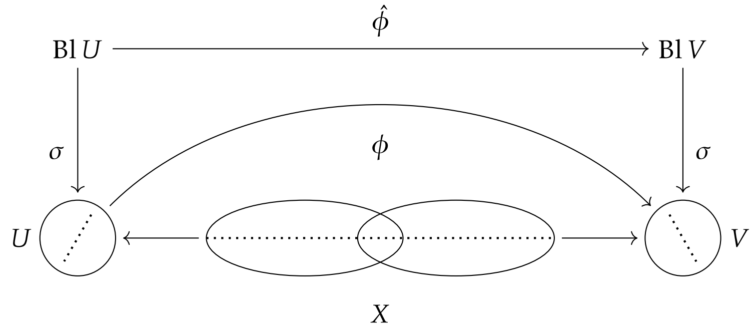

Figure 1: Blowups glue

We now calculate the canonical line bundle of the blowup at a point, where a point x gets replaced by $\mathbb{CP}^{n-1}$ and everything else stays the same:

Proposition 2.36 (Canonical line bundle of blowup): *Let E be the exceptional divisor in the blowup $\tilde{X} \to X$. Then*

$$K_{\tilde{X}} = \sigma^* K_X \otimes [(n-1)E]$$

Proof. Away from the exceptional divisor, everything is a bihilomorphism and $[E]$ is trivial, so we only need to check what happens on E. This is a local question, so we may as well assume $X = \mathbb{C}^n$. In local coordinates for the i−th patch, we have that $(l,z) \in Bl_0\,\mathbb{C}^n$ gets sent to $(\phi_j(l), z_i)$. O the other hand, the map σ sends $(l,z) \mapsto z$. Hence, in local coordinates σ looks like a swapping of z_i, together with $n-1$ multiplications by z_i:

$$(w_1, ..., w_{n-1}, z) \mapsto (zw_1, ..., z, ..., zw_{n-1})$$

If ω is a meromorphic $(n,0)$ form on X, i.e. a section of K_X, it looks locally like $f(z)dz_1 \wedge ... \wedge dz_n$. Therefore,

$$\sigma^*\omega = (f \circ \sigma)d(zv_1) \wedge ... \wedge dz \wedge ... \wedge d(zv_{n-1}) = z^{n-1}(f \circ \sigma)dv_1 \wedge ... \wedge dv_{n-1}$$

Hence, since $K_{\hat{X}} = (\sigma^*\omega)$ by 2.26, and outside E we just get $\sigma^* K_X$, whilst on E we pick up $n-1$ zeros, i.e. its order of vanishing along E is $n-1$, so $(\sigma^*\omega_E) = (n-1)[E]$. Note that E is an irreducible hypersurface, as it locally looks like the zero section $\mathbb{CP}^{n-1} \subset \mathcal{O}(-1)$!

If the canonical bundle does not admit any sections, we can do a similar cocycle computation on E that is a bit less neat. Basically, we compare the transition functions of projective space (first line) and the transition functions for the trivialization of $\hat{X}$ (second line)

$$(w_1, ..., w_n) \mapsto (\frac{w_1}{w_i}, ..., \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, ..., \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_j}{w_i}, ..., \frac{w_n}{w_i})$$
$$(u_1, ..., u_n) \mapsto (\frac{u_1}{u_i}, ..., \hat{1}, ... \frac{u_n}{u_i}, u_i u_{n+1})$$

We thus see that we have an extra u_i in the trivialization for projective space, and so when we take determinants of Jacobians to get the cocycle for the canonical bundles, we get an extra u_i giving us a cocycle $u_i^{-n+1}, u_i = z_i/z_j$ rather than u_i^{-n}, which was the cocycle for $K_{\mathbb{CP}^{n-1}} \simeq \mathcal{O}(-n)$. Notice also that $\mathcal{O}(E)$ has cocycles z_i/z_j as E is locally defined by $z_i = 0$. □

Corollary 2.37 (Line bundle on exceptional divisor): *For $E = \mathbb{CP}^{n-1} \subset \hat{X}$, one has*

$$\mathcal{O}(E)|_E \simeq \mathcal{O}(-1)$$

Proof. One approach is to use 2.28 and get

$$\mathcal{O}(nE) \simeq (K_{\hat{X}} \otimes \mathcal{O}(E))|_E \simeq K_E \simeq K_{\mathbb{CP}^{n-1}} \simeq \mathcal{O}(-n)$$

and then cancel out the n since the Picard group of projective space is torsion-free. Another approach is to do a cocycle computation, and yet another one is to realize $\mathcal{O}(E) \simeq \pi^*\mathcal{O}(-1)$ where $\pi : \mathrm{Bl}_0\, \mathbb{C}^n \to \mathbb{CP}^{n-1}$. □

2.7 Connections and characteristic classes using Chern-Weil theory

First Chern class of a line bundle, Kahler metrics and Levi-Civita connection, higher Chern classes.

2.7.1 Introduction: the first Chern class of a line bundle using cocycles and the Fubini-Study form

Let L be a holomorphic line bundle on X trivialized over U_i via holomorphic sections σ_i. The transition functions are just invertible holomorphic functions g_{ij} such that

$$\sigma_i = g_{ij}\sigma_j$$

on the overlaps. Given a hermitian inner product h on L, i.e. a smooothly varying inner product $L_x \times L_x \to \mathbb{C}$, we can evaluate it on σ_i and thus producing a positive function on U_i:

$$h_i = h(\sigma_i, \sigma_i) : U_i \to \mathbb{C}$$

We then have that $h_i = |g_{ij}|^2 h_j$ since $h(\lambda u, \lambda u) = |\lambda|^2 h(u,u)$. These functions now serve as the local Kähler potentials of a global two form which can be expressed locally as

$$\omega_i = \frac{1}{2i\pi}\partial\overline{\partial}\log h_i$$

These glue on the overlaps, as $\partial\overline{\partial}\log|g_{ij}|^2 = 0$. Up to $\frac{i}{2\pi}$, this is the curvature of the Chern connection, which we will describe in the next section. Importantly, ω is equal to $c_1(L)$ in cohomology. We now describe a standard example, the Fubini-Study form, which turns out to be exactly $c_1(\mathcal{O}(1))$ and hence is a generator of the cohomology of $\mathbb{CP}^n$.

Example (The Fubini-study form): Consider $\mathcal{O}(-1) \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$. Restricting the standard metric h on the trivial bundle over $\mathbb{CP}^n$ to $\mathcal{O}(-1)$ and then dualizing yields a Hermitian metric h^* on $\mathcal{O}(1)$. The 2-form then looks locally like

$$\omega_i = \frac{1}{2i\pi}\partial\overline{\partial}\log h_i^*$$

where $h_i^* = h^*(\sigma_i^*, \sigma_i^*) = \frac{1}{h(\sigma_i,\sigma_i)}$. On the standard opens U_i which trivialize $\mathcal{O}(-1)$, we seek the sections σ_i. This section inputs a line with $z_i \neq= 0$ and produces an element of the line. The most obvious thing we could do is take its normalization, which is also the triviaization, i.e. in local coordinates it looks like

$$\mathbb{C}^n \simeq U_i \to \mathcal{O}(-1)|_{U_i} \simeq U_i \times \mathbb{C} \simeq \mathbb{C}^{n+1}$$

$$(z_1, \ldots, \hat{1}, \ldots, z_n) \mapsto (z_1, \ldots, 1, \ldots, z_n)$$

Thus, we get an extra 1 and $h(\sigma_i, \sigma_i) = 1 + \sum |z_i|^2$ and hence

$$\omega_i = \frac{1}{2i\pi}\partial\overline{\partial}\log\left(\frac{1}{1+\sum|z_i|^2}\right) = \frac{-1}{2i\pi}\partial\overline{\partial}\log(1+\sum|z_i|^2) = \frac{i}{2\pi}\partial\overline{\partial}\log(1+\sum|z_i|^2)$$

We have not proved that this definition of the Chern class coincides with the one given in K-theory, where we defined $x = c_1(\mathcal{O}(-1))$ and showed that $\langle x^n, [\mathbb{CP}^{\ltimes}]\langle = (-1)^n$. Hence, we expect that $\omega = -x$ will evaluate the fundamental class to 1. We now give a detailed proof of this important fact.

Proposition 2.38 (Fubini-Study form is normalized):

$$\int_{\mathbb{CP}^n} \omega^n = 1$$

Proof. We have in local coordinates in an affine patch U, which is all that matters for integration, as it is dense and has complement a thin set which can be identified with $\mathbb{CP}^{n-1}$:

$$\omega = \frac{i}{2\pi}\left[\frac{\sum dz_j \wedge d\overline{z}_j}{1+\sum |z_j|^2} - \frac{(\sum \overline{z}_j \wedge dz_j)(\sum z_j \wedge d\overline{z}_j)}{(1+\sum |z_j|^2)^2}\right]$$

We can also write $Z = 1 + \sum |z_j|^2$ and hence

$$\omega = \frac{i}{2\pi}\frac{1}{Z^2}\sum(\delta_{ij}Z - \overline{z}_i z_j)dz_i \wedge d\overline{z}_j = \frac{i}{2\pi}\frac{1}{Z}\sum(\delta_{ij} - \frac{\overline{z}_i z_j}{Z})dz_i \wedge d\overline{z}_j$$

We describe the first two cases, to get an idea of what is happening:

- Case $n = 1$:

  The formula reduces to

  $$\omega = \frac{i}{2\pi}\frac{dz \wedge d\overline{z}}{(1+|z|^2)^2}$$

  In polar coordinates $z = re^{i\theta}$ we have that $dz \wedge d\overline{z} = -2ri\, dr \wedge d\theta$ and hence

  $$\int_{\mathbb{CP}^1} \omega = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{-2ri\, dr \wedge d\theta}{(1+r^2)^2} = \int_0^\infty \frac{d(r^2)}{(1+r^2)^2} = \int_1^\infty \frac{dt}{t^2} = 1$$

- Case $n = 2$:

  One gets the value

  $$\omega^2 = \frac{-1}{2\pi^2}\frac{dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2}{Z^3}$$

  Now we integrate and put polar coordinates again:

  $$\begin{aligned}\int_{\mathbb{CP}^2} \omega^2 &= \frac{-1}{2\pi^2}\int_{\mathbb{C}^2} \frac{(-2ir_1 dr_1 \wedge d\theta_1) \wedge (-2ir_2 dr_2 \wedge d\theta_2)}{(1+r_1^2+r_2^2)^3} \\ &= \frac{-2}{\pi^2}\int \frac{dr_1 \wedge dr_2 \wedge d\theta_1 \wedge d\theta_2}{(1+r_1^2+r_2^2)^3} = \\ &= \frac{(-2)(2\pi)^2}{4\pi^2}\int \frac{d(r_1)^2 \wedge d(r_2)^2}{(1+r_1^2+r_2^2)^3} = \\ &= -2\int_0^\infty \int_0^\infty \frac{dx \wedge dy}{(1+x+y)^3} = \\ &= -2\int_{y=0}^\infty [\frac{(1+x+y)^{-2}}{-2}|_{x=0}^\infty]dy = \\ &= \int_0^\infty \frac{dy}{(1+y)^2} = \int_1^\infty \frac{dt}{t^2} = 1\end{aligned}$$

The same thing in fact happens for all n. When we exponentiate, we get the following:

$$\omega^n = (\frac{i}{2\pi Z})^n [\sum_{i,j} (\delta_{ij} - \frac{\overline{z}_i z_j}{Z}) dz_i \wedge d\overline{z}_j]^n$$

We aim to show that we get $(\frac{i}{2\pi})^n \frac{n!}{Z^{n+1}} d\vec{z}$, where $d\vec{z} = dz_1 \wedge d\overline{z}_1 \wedge ... \wedge dz_n \wedge d\overline{z}_n$. This comes down to showing that

$$[\sum_{i,j} (\delta_{ij} - \frac{\overline{z}_i z_j}{Z}) dz_i \wedge d\overline{z}_j]^n = \frac{n!}{Z} d\vec{z}$$

An arbitrary term of this look like

$$[\prod_{k=1}^{n} \delta_{\sigma k, \tau k} - \frac{\overline{z}_{\sigma k} z_{\tau k}}{Z}] dz_{\sigma 1} \wedge d\overline{z}_{\tau 1} \wedge ... \wedge dz_{\sigma n} \wedge d\overline{z}_{\tau n}$$

since we cannot have repetitions in any dz_i, as $dz_i \wedge dz_i = 0$.

We think of this as a polynomial in $1/Z$, and get terms of the form

$$\pm \delta_{i_1, j_1} ... \delta_{i_k, j_k} \frac{\overline{z}_{i_{k+1}} z_{j_{k+1}}}{Z} ... \frac{\overline{z}_{i_n} z_{j_n}}{Z} dz_{i_1} \wedge d\overline{z}_{j_1} \wedge ...$$

Here, i_k, j_k are permutations, and we are picking up only one two form from every term in the product.

Clearly, the term of Z^0 is given by multiplying all the $\delta_{i,j}$, so we are summing over all things that look like

$$\prod_{k=1}^{n} dz_{i_k} \wedge d\overline{z}_{i_k}$$

for $\sigma = \{i_1, ..., i_n\}$ some permutation of $\{1, 2, ..., n\}$. To convert this into the form $d\vec{z}$, an even number of interchanges is done, so the sign doesnt change and all of these $n!$ terms are equal to $d\vec{z}$. This gives $n! d\vec{z}$.

Now, for the coefficient of $\frac{1}{Z}$, we are summing over all terms where we are multiplying $n-1$ δ's,together with a single $\frac{\overline{z}_i z_j}{Z}$. The δ's force every pair to be equal: $i_k = j_k$, otherwise we would get 0, and so $i_1 = j_1, ..., i_{n-1} = j_{n-1}$, forces $i_n = j_n$. Hence, we get terms of the form

$$\frac{-z_{\sigma k} \overline{z}_{\sigma k}}{Z} dz_{\sigma 1} \wedge d\overline{z}_{\sigma 1} \wedge ...$$

for any permutation σ. Again, the form part is $d\vec{z}$, since we have an even number of interchanges, and hence we get all in all:

$$-n! \frac{z_1 \overline{z}_1 + ... + z_n \overline{z}_n}{Z} d\vec{z} = -n! \frac{Z-1}{Z} d\vec{z}$$

When we add the Z^0 and Z^{-1} terms we get precisely

$$n! d\vec{z} - n! \frac{Z-1}{Z} d\vec{z} = \frac{n!}{Z} d\vec{z}$$

This is what we want, and so we show that all the higher coefficients of $Z^{-m}, m > 1$ vanish!

To do that, note that the odd and even permutations come in pairs and for any k, any transposition $T \in S_k$ gives a bijection between the two sets, i.e. the odd and even permutations are two cosets with the same size. Hence, for any k

$$\sum_{\sigma \in S_k} (-1)^{\operatorname{sgn} \sigma} = 0$$

This combinatorial identity will show that all the terms for a higher power Z^{-m}, when we are not forced to have all $i_k = j_k$, come in pairs of opposite signs and cancel out. The crucial part is that now $m \geq 2$ so we get terms where we are actually allowed transpositions!

These higher order terms will have the schematic form

$$\frac{(-1)^m}{Z^m} \underbrace{\delta ... \delta}_{n-m} \underbrace{\overline{z}z ... \overline{z}z}_{m} \pm d\vec{z}$$

The sign $\pm$ is important, as it will lead to cancellations. Let B be a subset of $\{1, ..., n\}$ such that $|B| = m$. For this, we get terms

$$[\prod_{a \in B^c} \delta_{\sigma a, \tau a} \prod_{b \in B} \overline{z}_{\sigma b} z_{\tau b}] dz_{\sigma 1} \wedge d\overline{z}_{\sigma 1} \wedge ...$$

For this to be nonzero, we must have that $\sigma a = \tau a, a \notin B$. We can ignore the forms $dz_{\sigma a} \wedge d\overline{z}_{\tau a}$ for the purpose of determining the sign, for the same reason as before that the sign to convert a term like this to dvol will be 1. The rest looks like this:

$$\prod_{b \in B} \left[\overline{z}_{\sigma b} z_{\tau b} \, dz_{\sigma b} \wedge d\overline{z}_{\sigma b} \right]$$

Since $\sigma a = \tau a, a \notin B$, putting $\gamma = \tau \circ \sigma^{-1}$ we can write this as

$$\mathfrak{B}_\gamma := \prod_{c \in \sigma B} \left[\overline{z}_c z_{\gamma c} \, dz_c \wedge d\overline{z}_{\gamma c} \right]$$

We know that $\gamma c = c$ for $c \notin \sigma B$, and we can thus think of it as a permutation of σB. Moreover, the only other terms having the same $\overline{z}z$ coefficient will have to come from another permutation of σB like γ. But now $|B| = m > 1$, hence we can apply a transposition! For any such σ, τ which give us γ, there is a transposed $\gamma' = T \circ \gamma$ which gives the opposite sign, as if we transpose $d\overline{z}_{\gamma c}$ and $d\overline{z}_{\gamma c'}$, $c' > c$ we have to do a total of $2(c' - c) + 2(c' - c) - 1$ moves to get back to the form that γ gives. In other words,

$$\mathfrak{B}_{T \circ \gamma} = -\mathfrak{B}_\gamma$$

and hence everything cancels out.

For example, to get to

$$dz_1 \wedge d\overline{z}_{\gamma_1} \wedge dz_2 \wedge d\overline{z}_{\gamma_2} \wedge dz_3 \wedge d\overline{z}_{\gamma_3}$$

from the transposed version

$$dz_1 \wedge d\overline{z}_{\gamma_3} \wedge dz_2 \wedge d\overline{z}_{\gamma_2} \wedge dz_3 \wedge d\overline{z}_{\gamma_1}$$

we need to move the $d\overline{z}_{\gamma_1}$ 4 times to the left and the $d\overline{z}_{\gamma_3}$ 3 times to the right, resulting in a negative sign, since $2.(3-1) + 2(3-1) - 1 = 7$ is odd.

All in all, this shows that

$$\omega^n = (\frac{i}{2\pi})^n \frac{n!}{Z^{n+1}} d\vec{z}$$

We can integrate, do polar coordinates again and thus get an iterated integral using Fubini:

$$\int_{\mathbb{CP}^n} \omega^n = \frac{i^n n!}{(2\pi)^n} \int_{\mathbb{C}^n} \frac{d\vec{z}}{Z^{n+1}} =$$

$$= \frac{i^n n!(-i)^n}{(2\pi)^n} \int_{\mathbb{C}^n} \frac{d(r_1)^2 \wedge d\theta_1 \wedge ... \wedge d(r_n^2) \wedge d\theta_n}{(1+r_1^2+...+r_n^2)^{n+1}} =$$

$$= \frac{n!}{(2\pi)^n} \int_{\mathbb{C}^n} \frac{(-1)^{n-1} d(r_1^2) \wedge ... \wedge d(r_n^2) \wedge d\theta_1 \wedge ... \wedge d\theta_n}{(1+r_1^2+...+r_n^2)^{n+1}} =$$

$$= (-1)^{n-1} n! \int_{x_i=0}^{\infty} \frac{dx_1 \wedge dx_2 ... \wedge dx_n}{(1+x_1+...+x_n)^{n+1}} =$$

$$= (-1)^{n-1} n! [\int_0^\infty ...[\int_0^\infty \frac{dx_1}{(1+x_1+...+x_n)^{n+1}}] \wedge dx_2] ... \wedge dx_n]$$

$$= \frac{(-1)^{n-1} n!}{(-n)(-(n-1))...(-2)} \int_0^\infty \frac{dx_n}{(1+x_n)^2} = 1$$

This proves the normalization identity for all n. $\square$

2.7.2 The First Chern class using connections, i.e. baby Chern-Weil theory

Given a holomorphic line bundle $L \to X$, we are interested in the connections on it, i.e. the maps

$$\Gamma(L) \xrightarrow{d^{\mathcal{A}}} \Gamma(T^*X \otimes L)$$

satisfying the Leibniz rule. Recall the local expression of this:

$$d^{\mathcal{A}_\alpha} s_\alpha = ds_\alpha + A_\alpha s_\alpha$$

Here, A_α is a complex one-form in $\Omega^1(U_\alpha)^{\mathbb{C}}$. The transformation law, reads

$$A_\beta = A_\alpha + \psi_{\beta\alpha} d\psi_{\beta\alpha}^{-1}$$

since, in the 1-dimensional case, conjugating A_α leaves it the same. The curvature $F(\mathcal{A})$ is, as usual, the square of this differential operator, which is an $End(L)$-valued 2-form on X and locally is expressed as

$$F_\alpha = dA_\alpha$$

since $A \wedge A = 0$, since we're working with line bundles. In other words,

$$Fs = (d+A)^2 s = d^2 s + d(As) + A(ds) + A \wedge As$$

But $d^2 = A \wedge A = 0$ and $d(As) = (dA)s - A(ds)$ by Leibniz, hence all we get is $(dA)s$. Recall that the difference between any two connections is a one-form a, and thus the difference between any two curvature forms is an exact form da. Hence, we can associate to L the cohomology class

$$[F] \in H^2(X;\mathbb{C}) = H^2_{dR}(X) \otimes \mathbb{C}$$

Recall that a connection is compatible with the metric if

$$d\langle s, s' \rangle = \langle d^{\mathcal{A}} s, s' \rangle + \langle s, d^{\mathcal{A}} s' \rangle$$

In the case of a unitary i.p., any such connection is purely imaginary. Hence, this allows the following definition.

Definition 2.39 (First Chern class of a line bundle): *Given a unitary connection $\mathcal{A}$ on a line bundle L over X, we define*

$$c_1(L) := [\frac{i}{2\pi}F] \in H^2_{dR}(X)$$

Proposition 2.40 (Chern class of tensor product):

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

Proof. We define a tensor product connection $\mathcal{A} \otimes \mathcal{A}'$ by the Leibniz rule and get

$$d^{\mathcal{A}\otimes\mathcal{A}'}(s \otimes s') = d^{\mathcal{A}} s \otimes s' + s \otimes d^{\mathcal{A}'} s'$$

In our case of line bundles, we have that this is locally just

$$(ds + As)s' + s(ds' + A's') = d(ss') + (A + A')ss'$$

This shows that the curvature of this induced connection is just

$$F(\mathcal{A} \otimes \mathcal{A}') = F(\mathcal{A}) + F(\mathcal{A}')$$

□

2.7.3 The Chern connection

Definition 2.41 (Chern connection on line bundles): *There is a unique connection which is compatible with any chosen Hermitian inner product and whose local 1-forms are holomorphic.*

Proof. Locally, we have a Hermitian norm $h : U \to \mathbb{R}^+$ and a smooth section s of L is just a smooth function $\lambda : U \to \mathbb{C}$. We have that

$$d\langle s, s\rangle = d(h(\lambda, \lambda)) = d(\lambda\overline{\lambda}h) = (d\lambda)\overline{\lambda}h + (d\overline{\lambda})\lambda h + (dh)\lambda\overline{\lambda}$$

We require that this is the same as

$$\langle d^{\mathcal{A}}s, s\rangle + \langle s, d^{\mathcal{A}}s\rangle = \langle ds + As, s\rangle + \langle s, ds + As\rangle = h\overline{\lambda}\, d\lambda + h\lambda\, d\overline{\lambda} + h\lambda\overline{\lambda}(A + \overline{A})$$

For these to match up we must have that $A + \overline{A} = h^{-1}\, dh$ and hence the real part is

$$A^{1,0} = h^{-1}\,\partial h = \partial(\log h)$$

This shows uniqueness. For existence, we define the connection locally by the above and show it glues. Well, given holomorphic transition ψ we have that the Hermitian i.p. transforms as $h' = \psi\overline{\psi}h$, as it is a section of $L \otimes L^\vee$. As ψ is holomorphic, we have that $\partial\overline{\psi} = 0$ and $\partial\psi = d\psi$. Then

$$A' = \partial \log h' = \partial \log(\psi\overline{\psi}h) = \partial \log(h) + \partial \log(\psi\overline{\psi}) =$$
$$= A + \frac{\partial(\psi\overline{\psi})}{\psi\overline{\psi}} = A + \frac{\partial\psi}{\psi} = A + \psi^{-1} d\psi$$

showing it transforms correctly. □

Corollary 2.42 (Curvature of Chern connection): *As a corollary, we have that the curvature of the Chern connection is*

$$F(\mathcal{A}) = d(\partial \log h) = \overline{\partial}\partial \log h = \frac{i}{2} d d^c \log |s|^2$$

where s is any nonvanishing holo section. Moreover,

$$[\frac{i}{2\pi} F(\mathcal{A})] = c_1(L) \in H^{1,1}(X)$$

Note that we insert $\frac{i}{2\pi}$ as a normalization factor so that, for example, 2.38 holds.

2.7.4 Poincare duality and the first Chern class of a divisor

Recall that, for an analytic hypersurface $Y \subset X$ in a compact complex manifold, the integration pairing sets up a duality and there is a unique form $\eta_Y \in H^2_{dR}(X)$ such that

$$\int_{Y^*} \varphi = \int_X \varphi \wedge \eta_Y$$

for any closed test form φ. We have the following:

Theorem 2.43 (Chern class of associated line bundle is Poincare dual to divisor): *Given a divisor $D = \sum a_i Y_i$, we define its Poincare dual as $\eta_D = \sum a_i \eta_{Y_i}$. Then we have that*

$$\eta_D = c_1([D])$$

i.e.

$$\frac{-1}{2\pi i} \int_X F(\mathcal{A}) \wedge \varphi = \sum a_i \int_{Y^*} \varphi$$

Proof. We will prove this by first choosing our connection to be the Chern connection and reduce to the case $D = Y$ an analytic hypersurface. Let's say we have local defining functions f_α on an open cover U_α, which gives us a holomorphic section s of $[Y]$ (recall that transition f-n's for $[Y]$ are f_β/f_α, so the f_α' glue together). As a divisor, $Y = (s)$. We have a little tubular neighbourhood of Y given by all $x \in X$ such that $|s(x)| \leq \epsilon$. We denote its complement by $X(\epsilon)$. Thus

$$\int_X F(\mathcal{A}) \wedge \varphi = \frac{i}{2} \int_X d d^c \log h \wedge \varphi = \lim_{\epsilon \to 0} \frac{i}{2} \int_{X(\epsilon)} d d^c \log |s|^2 \wedge \varphi =$$

$$\overset{Stokes}{=} - \lim_{\epsilon \to 0} \frac{i}{2} \int_{\partial X(\epsilon)} d^c \log |s|^2 \wedge \varphi$$

On local neighbourhoods U_α, we have that

$$d^c \log |s|^2 = i(\partial - \overline{\partial}) \log(f_\alpha \overline{f_\alpha} h_\alpha) = i(\overline{\partial} \log \overline{f_\alpha} - \partial \log f_\alpha + (\overline{\partial} - \partial) \log h_\alpha))$$

However, the integral over $\log h_\alpha$ should tend to 0. Moreover, the integral over the conjugated f_α is just the conjugate of the integral over f_α. We thus get that our integral is equal to

$$- \lim_{\epsilon \to 0} \frac{i}{2} \int_{\partial X(\epsilon) \cap U_\alpha} d^c \log |s|^2 \wedge \varphi = - \lim_{\epsilon \to 0} i \operatorname{Im} \int_{\partial X(\epsilon) \cap U_\alpha} \partial \log f_\alpha \wedge \varphi$$

Now let's extend f_α to a local coordinate system $(z_1 = f_\alpha, z_2, \ldots, z_n)$, where $(z_2, \ldots, z_n)$ give local coordinates for $Y \cap U_\alpha$. Moreover, let $\varphi = \tilde{\varphi} + \varphi_1$ where $\tilde{\varphi}$ collects all the dz_1 and $d\tilde{z_1}$ terms. We use the residue theorem to conclude that

$$\int_X F(\mathcal{A}) \wedge \varphi = -i \operatorname{Im} \lim_{\epsilon \to 0} \int_{|z_1| = \epsilon/\sqrt{h_\alpha}} \frac{dz_1}{z_1} \wedge \varphi_1 = -2i\pi \int_{z_1 = 0} \varphi_1(0, z_2, \ldots, z_n)$$

By patching together, we are done. □

Remark (Relationship with Euler class): The reason all of this works out is that for a hypersurface Y, $\mathcal{N}_{Y/X} \simeq \mathcal{O}_X(Y)|_Y$. On the other hand, $pd(Y) = j^*(\iota^*)^{-1} u_{\mathcal{N}}$, which should just give $s_0^* j^* u_{\mathcal{O}_X(Y)} = e(\mathcal{O}_X(Y)) = c_1(Y)$.

Remark (Intersection pairing): By Poincare duality, on a complex surface S the intersection pairing is given by

$$\#(M_1 \cap M_2) = \int_S c_1(M_1) \wedge c_1(M_2)$$

Proposition 2.44 (Self-intersection of exceptional divisor): *Let X be a compact complex surface. Then the exceptional divisor in the blowup satisfies $E \cdot E = -1$.*

Proof. We know by 2.37 that $\mathcal{O}(E)|_E \simeq \mathcal{O}(-1)$. Now we use Poincare duality

$$E \cdot E = \int_{\tilde{X}} c_1(\mathcal{O}(E)) \wedge c_1(\mathcal{O}(E)) = \int_E c_1(\mathcal{O}(E)) = \int_{\mathbb{CP}^1} c_1(\mathcal{O}(-1)) = -1$$

□

Example (Riemann surfaces): For a Riemann surface X, define the degree of a line bundle to be

$$\deg L = \int_X c_1(L)$$

We thus have

$$\deg[D] = \langle c_1([D]), [X] \rangle = \langle \eta_D, [X] \rangle = \langle \sum a_i \eta_{Y_i}, [X] \rangle = \sum a_i = \deg D$$

On the right hand side we have the degree of D as a divisor. In other words, the first Chern class of the associated line bundle corresponds to the integer $\deg D$. For example, on the Riemann sphere, $\deg \mathcal{O}(k) = k$. We want to show these are in fact all the holo line bundles over $\mathbb{CP}^1$, i.e. $\operatorname{Pic}(\mathbb{CP}^1) = \mathbb{Z}$.

Proposition 2.45 (Chern class detects triviality): *If L is a holo line bundle over $\mathbb{CP}^1$ with $c_1(L) = 0$ then L is holomorphically trivial.*

Proof. The first Chern class is given by the curvature of any connection on L. This means all connections have an exact curvature form, so we can subtract such an exact form and choose a flat connection $\mathcal{A}$ on L, which means that locally $dA_\alpha = 0$. By the usual smooth Poincare lemma, we know these are exact, hence $A_\alpha = da_\alpha$. Put $\psi = e^{-a_\alpha}$; then

$$A_\alpha + \psi^{-1} d\psi = 0$$

The Riemann sphere has the usual clutching trivialization U_0, U_1. WLOG $A_0 = A_1 = 0$, meaning that the transition function ψ_{01} is constant on the intersection $U_0 \cap U_1$. This means we can glue the local trivs to a global C^∞ one, which we need to show is holomorphic as well. To do this, we need to find a nonvanishing global holo section. We already know that $L \simeq \mathbb{CP}^1 \times \mathbb{C}$ as smooth manifolds. Choose a Hermitian i.p. and use the Chern connection $\mathcal{A}$. A section s is holomorphic if and only if

$$\overline{\partial^{\mathcal{A}}} s = 0, d^{\mathcal{A}} = \partial^{\mathcal{A}} + \overline{\partial^{\mathcal{A}}}$$

To find a holomorphic section we will try to find a function $f : \mathbb{CP}^1 \to \mathbb{C}$ such that e^f is a global nonvanishing holo section. The equation becomes

$$0 = \overline{\partial^{\mathcal{A}}} s = \overline{\partial} s + A'' s \iff \overline{\partial} f = -A'', s = e^f$$

We want to show this always has a solution. On the open cover $U_0 \cup U_1$, the Poincare lemma tells us that there exists solutions f_0, f_1 agreeing on the intersection $\mathbb{C}^\times$, i.e.

$$\overline{\partial}(f_1 - f_0) = 0$$

Hence, we may Laurent expand:

$$f_1 - f_0 = \sum_{n \in \mathbb{Z}} c_n z^n$$

This allows us to define a holomorphic f by

$$f = \begin{cases} f_0 + \sum_{n=0}^{\infty} c_n z^n \text{ on } U_0 \\ f_1 - \sum_{n<0} c_n z^n \text{ on } U_1 \end{cases}$$

□

Remark: this can also be seen from the Hodge decomposition, i.e. $H^1(S^2) = 0 \implies H^{0,1}(S^2) = 0$.

2.7.5 Connections on complex vector bundles

A connection on a vector bundle is a differential operator that takes a section of E and outputs an E-valued 1-form, satisfying the Leibniz rule. It extends to a map

$$D : C^\infty(X, E \otimes \Lambda^k \Omega) \to C^\infty(X, E \otimes \Lambda^{k+1} \Omega)$$

that satisfies a graded Leibniz rule. If we trivialize E in a neighbourhood by sections $e_1, ..., e_r$, then an $E-$valued k form has the form $\sigma = s_1 \otimes e_1 + ... + s_r \otimes e_r$ for some k-forms s_i. The Leibniz rule then shows that

$$D\sigma = \sum ds_i \otimes e_i + (-1)^k s_i \otimes De_i$$

So all we need to know is what De_i is, which can be expressed in terms of a matrix of one-forms $A = (a_{ij})$

$$De_j = \sum a_{ij} \otimes e_i$$

Hence, we can rewrite D in a local trivialization where $\sigma = s = (s_1, ..., s_r)$

$$D\sigma = ds + AS$$

The transformation rule is as follows: if g transforms from the second trivialization to the second, then

$$s = gs', D(s) = gD(s')$$

then $ds = dg\,s' + g\,ds' = g(g^{-1}dg\,s' + ds')$ and hence

$$D(s) = ds + As = g(g^{-1}dg\,s' + ds') + Ags' = g(ds' + (g^{-1}dg + g^{-1}Ag)s') = gD(s') = g(ds' + A's')$$

and hence

$$A' = g^{-1}dg + g^{-1}Ag$$

Applying D twice we see that the curvature is given by

$$D^2 s = (dA + A \wedge A)s$$

Definition 2.46 (Compatibility with Hermitian metric): *We say a connection D is compatible with a Hermitian metric on the complex vector bundle E if*

$$d\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^p \{\sigma, D\tau\}$$

for any $\sigma \in C^\infty(X, \omega_{X,\mathbb{C}})^p \otimes E), \tau \in C^\infty(X, \omega_{X,\mathbb{C}})^q \otimes E)$ where $\{\sigma, \tau\}$ is the hermitian product on the E part and wedging on the form part.

Proposition 2.47 (Local matrix of a compatible connection): *A connection D is compatible with h if and only if in an isometric trivialization e_i such that $h(e_i, e_j) = \delta_{ij}$ we have*

$$\overline{A}^T = -A$$

Proof. We have $\{\sigma, \tau\} = \sigma^T \wedge \overline{\tau}$. Now we can apply the Leibniz rule

$$d\{\sigma, \tau\} = d\sigma^T \wedge \overline{\tau} + (-1)^p \sigma^T \wedge d\overline{\tau}$$

On the other hand, $D = d + A$ so

$$\{D\sigma, \tau\} = (d\sigma + A\sigma)^T \wedge \overline{\tau} = d\sigma^T \wedge \overline{\tau} + (-1)^p \sigma^T \wedge A^T \wedge \overline{\tau}$$

and

$$\{\sigma, D\tau\} = \sigma^T \wedge (\overline{d\tau + A\tau}) = \sigma^T \wedge d\overline{\tau} + \sigma^T \wedge \overline{A} \wedge \overline{\tau}$$

Hence,

$$\{D\sigma, \tau\} + (-1)^p \{\sigma, D\tau\} - d\{\sigma, \tau\} = (-1)^p \sigma^T \wedge (A^T + \overline{A}) \wedge \overline{\tau}$$

□

Note that if H denotes the matrixof the inner product, then for an arbitrary frame we have $dH = A^T H + H\overline{A}$. If A is compatible, then this can be thought of as a commutator.

2.7.6 Connections on holomorphic vector bundles

Let now X be a complex manifold. By projecting $\Omega^1_{X,\mathbb{C}}$ onto the $(1,0)$ and $(0,1)$ parts, we get operators $D^{1,0}, D^{0,1}$ which locally behave as

$$D^{1,0}s = \partial s + A^{1,0} \wedge s$$
$$D^{0,1}s = \overline{\partial} s + A^{0,1} \wedge s$$

They also satisfy a Leibniz rule. If E is a holomorphic vector bundle, then the transition matrices are holomorphic and we can define $\overline{\partial}_E$ locally by applying $\overline{\partial}$ to each entry in the trivialization, and this will glue by the Leibniz rule as $\overline{\partial}$ applied to a holomorphic matric vanishes, i.e. $\overline{\partial}_E(s') = \overline{\partial}_E(Ms) = \overline{\partial}(M)s + M\overline{\partial}_E s = +M\overline{\partial}_E s$.

Definition 2.48 (Chern connection): *There is a unique connection on any Hermitian holomorphic vector bundle (E,h) such that $D^{0,1} = \overline{\partial}_E$.*

Proof. If such a thing exists, it will kill off any holomorphic sections, and hence in the local formula $D^{0,1} = \overline{\partial} + A^{0,1}$ we must have $A^{0,1} = 0$ and so A is a matrix of $(1,0)$-forms. Now, we have by compatibility that

$$dH_{ij} = d\langle e_i, e_j\rangle = \langle De_i, e_j\rangle + \langle e_i, De_j\rangle = \langle a_{ki}e_k, e_j\rangle + \langle e_i, a_{kj}e_k\rangle = a_{ki}H_{kj} + \overline{a_{kj}}H_{ik}$$

where H is the matrix for the inner product. In other words,

$$dH = A^T H + H\overline{A}$$

Since A consists of only $(1,0)$-forms and H is a matrix of functions, we can compare the $(0,1)$ part to conclude that

$$\overline{\partial} H = H\overline{A} \implies A = \overline{H}^{-1}\partial\overline{H}$$

Now, the Hermirian inner product is a section of $\mathrm{Hom}(E \otimes \overline{E} \to \mathbb{C}) = \mathrm{Hom}(\overline{E}, E^\vee)$ which should transfrom as in the schematic diagram

$$\begin{tikzcd}
\overline{E} \arrow[r, "H'"] \arrow[d, "\overline{g}"'] & E^\vee \\
\overline{E} \arrow[r, "H"'] & E^\vee \arrow[u, "g^T"']
\end{tikzcd}$$

I.e. $H' = g^T H \overline{g}$. We now check that A transforms correctly, i.e. is coherent with the transformation rule $A' = g^{-1}dg + g^{-1}Ag$. But g is holomorphic, so $0 = \overline{\partial} g = \partial\overline{g}$. Hence,

$$\overline{H'}^{-1}\partial\overline{H'} = (\overline{g}^T\overline{H}g)^{-1}\partial(\overline{g}^T\overline{H}g) = (\overline{g}^T\overline{H}g)^{-1}[\partial\overline{g}^T\,\overline{H}g + \overline{g}^T\partial\overline{H}g + \overline{g}^T\overline{H}\partial g] =$$
$$= (\overline{g}^T\overline{H}g)^{-1}[\overline{g}^T(\partial\overline{H}g + \overline{H}\partial g)] = g^{-1}\overline{H}^{-1}(\partial\overline{H}g + \overline{H}\partial g) = g^{-1}Ag + g^{-1}\partial g = g^{-1}Ag + g^{-1}dg = A'$$

□

Corollary 2.49 (Corollary): *For the Chern connection D we have that $\partial A = -A \wedge A$ and hence the curvature $dA + A \wedge A = \overline{\partial} A$ is of type $(1,1)$, and moreover $\overline{\partial}$–exact.*

Proof. Expand the equation $\partial M^{-1} = -M^{-1} \partial M M^{-1}$ for $M = \overline{H}$ and use the defining identity for A. ☐

2.7.7 Revisiting line bundles

When L is a line bundle, the Hermitian matrix is given by $\langle e_1, e_1 \rangle = \|e_1\|^2$ for some trivializing section e_1. Hence, we hacn write this as a function $H = e^{-\varphi}$ and then $A = \overline{H}^{-1} \partial \overline{H} = -\partial \varphi$ which shows that the curvature is simply $\overline{\partial} A = \partial \overline{\partial} \varphi$. Hence,

$$\frac{i}{2\pi} F(A) = \frac{1}{2\pi i} \partial \overline{\partial} \log \|s\|^2$$

Definition 2.50 (Positivity): *L is positive if it admits a metric h whose Chern curvature form defines a Hermitian product on T_X. In other words, if h is given by a weight φ, this is equivalent to*

$$F = \partial \overline{\partial} \varphi(z) = \sum \frac{\partial^2 \varphi}{\partial z_j \overline{\partial} z_k} dz_j \wedge d\overline{z}_k$$

having $(\frac{\partial^2 \varphi}{\partial z_j \overline{\partial} z_k})$ positive definite. An example is $\mathcal{O}(1)$ on $\mathbb{CP}^n$ which products the Fubini-Study metric.

2.7.8 The relationship between the Chern and Levi-Civita connection

Flatness of J

Note that, by definition of an induced connection, we should get the Leibniz rule to define ∇J implicitly as a section of $End(TX)$:

$$\nabla(J, v) = (\nabla J, v) + (J, \nabla v)$$

Hence, $\nabla J = 0$ precisely when $J(\nabla v) = \nabla(Jv)$, i.e. J commutes with ∇. Assuming this holds, let's see what happens:

Now, we apply this to the fundamental form $\omega = g(J-,-)$ and see that the induced connection must satisfy

$$\begin{aligned}
d(g(JX, Y)) = \nabla(g(JX, Y)) = \nabla(\omega(X, Y)) = \\
=^{dfn} (\nabla \omega)(X, Y) + \omega(\nabla X, Y) + \omega(X, \nabla Y) = \\
= (\nabla \omega)(X, Y) + g(J\nabla X, Y) + g(JX, \nabla Y) = \\
= (\nabla \omega)(X, Y) + g(\nabla JX, Y) + g(JX, \nabla Y) = \\
=^{\text{Levi-Civita compatible with g}} (\nabla \omega)(X, Y) + d(g(JX, Y))
\end{aligned}$$

Hence, if $\nabla J = 0$ for the Levi-Civita connection, we must have that $\nabla\omega = 0$. In fact, this is an if and only if, since this computation tells us that

$$(\nabla\omega)(X,Y) = g(\nabla JX - J\nabla X, Y)$$

So for the Levi-Civita connection, $\nabla J = 0 \iff \nabla\omega = 0$.

Expressing d using ∇

Now consider an arbitrary connection ∇ and the formula for the exterior derivative

$$(d\omega)(X,Y,Z) = X\cdot\omega(Y,Z) - Y\cdot\omega(X,Z) + Z\cdot\omega(X,Y) -$$

$$\omega([X,Y],Z) - \omega([Y,Z],X) + \omega([X,Z],Y)$$

We use the torsion

$$T(A,B) = \nabla_A B - \nabla_B A - [A,B]$$

The previous calculation showed that

$$(\nabla\omega)(A,B) = d(\omega(A,B)) - \omega(\nabla A,B) - \omega(A,\nabla B)$$

Contracting, we get

$$\iota_C(\nabla\omega)(A,B) = C\cdot(\omega(A,B)) - \omega(\nabla_C A,B) - \omega(A,\nabla_C B)$$

We can now replace each $[A,B]$ with $\nabla_A B - \nabla_B A - T(A,B)$ to get:

$$(d\omega)(X,Y,Z) = X\cdot\omega(Y,Z) - Y\cdot\omega(X,Z) + Z\cdot\omega(X,Y) -$$

$$\omega(\nabla_X Y - \nabla_Y X - T(X,Y),Z) - \omega(\nabla_Y Z - \nabla_Z Y - T(Y,Z),X) +$$

$$\omega(\nabla_X Z - \nabla_Z X - T(X,Z),Y)$$

Using the above formula for $\nabla\omega$, this converts to

$$d\omega(X,Y,Z) = [X\cdot\omega(Y,Z) - \omega(\nabla_X Y,Z) - \omega(Y,\nabla_X Z)]$$

$$-[Y\cdot\omega(X,Z) - \omega(\nabla_Y X,Z) - \omega(X,\nabla_Y Z)]$$

$$+[Z\cdot\omega(X,Y) - \omega(\nabla_Z X,Y) - \omega(X,\nabla_Z Y)]$$

$$+\omega(T(X,Y),Z) + \omega(T(Y,Z),X) - \omega(T(X,Z),Y) =$$

$$= \iota_X\nabla\omega(Y,Z) - \iota_Y\nabla\omega(X,Z) + \iota_Z\nabla\omega(X,Y)$$

$$+torsion$$

In particular, for the Levi-Civita connection, the torsion term is 0 and hence $\nabla\omega = 0 \implies d\omega = 0$.

Relating the connectons

We would like to relate the Chern connection $\mathfrak{C}$ on $T^{1,0}X$ and the Levi-Civita connection ∇ on TX, which are isomorphic as real bundles.

If $\mathfrak{C} = \nabla$, identified using the isomorphism, then since $\mathfrak{C}$ is $\mathbb{C}$-linear and the isomorphism identifies i with J, then ∇ would be J-linear, i.e. $i(\mathfrak{C}-) = \mathfrak{C}(i-)$ gets converted to $J(\nabla-) = \nabla(J-)$, and hence $\nabla J = 0$ which then implies that $\nabla\omega = 0$ and finally $d\omega = 0$, since ∇ is torsion-free.

Conversely, supposing that ω is Kähler i.e. $d\omega = 0$, then we can check this locally? The Levi-Civita and Chern connections coincide on $\mathbb{C}^n$ and since any Kähler metric osculates to the standard one up to second order, then the connections would have to coincide, since the matrices of these connections depend only on the metrics up to first order (hmm...why?)

2.8 Kähler geometry, del del bar lemma, Fubini-study metric

In this section we introduce a large class of complex manifolds for which we will prove the Hodge decomposition, namely the Kähler manifolds.

2.8.1 Hermitian structures and Kähler forms - linear algebra

Let V be a complex vector space and $V_{\mathbb{R}}$ be the underlying real vector space and $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ its complexification. Further, let $W = \mathrm{Hom}(V, \mathbb{R})$ be the dual of V and define $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$ similarly. These vector spaces have an almost complex structure and split as before:

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}, W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$$

The second exterior power splits as

$$\bigwedge^2 W_{\mathbb{C}} = \bigwedge^2 W^{1,0} \oplus (W^{1,0} \wedge W^{0,1}) \oplus \bigwedge^2 W^{0,1}$$

We denote the middle part as $W^{1,1}$ and by $W^{1,1}_{\mathbb{R}}$ its real part.

A Hermitian form on V is a map $V \times V \to \mathbb{C}$ that is $\mathbb{C}$-linear in the first variable, $\mathbb{C}$-antilinear in the second and satisfies $h(u,v) = \overline{h(v,u)}$. There is a correspondence between real (1,1)-forms and Hermitian forms, as the following proposition shows.

Proposition 2.51 (Hermitian forms and 1-1 forms): *There is a bijection between Hermitian forms on V and real (1,1) forms on V given by $h \mapsto -Im(h)$ and $\omega \mapsto h$ with $h(u,v) = \omega(u, Jv) - i\omega(u,v)$*

Proof. Since h is Hermitian, we see that $\omega = -Im(h)$ is alternating. It is of type $(1,1)$ since $\omega(u - iJu, v - iJv) = 0$ (since h is Hermitian) i.e. it vanishes on pairs of elements of $V^{1,0}$ and similarly with $V^{0,1}$.

To show the converse, put $h(u,v) = \omega(u,Jv) - i\omega(u,v)$. Then $h(u,Jv) = \omega(u,-v) - i\omega(u,Jv) = -\omega(u,v) - i\omega(u,Jv) = -ih(u,v)$. Since ω is of type $(1,1)$, we have that $\omega(u,Jv) = -\omega(Ju,v)$ using the fact that $0 = \omega(u - iJu, v - iJv) = [\omega(u,v) - \omega(Ju,Jv)] - i[\omega(u,Jv) + \omega(Ju,v)]$ and looking at the imaginary part. Now, using the fact that ω is alternating, we have

$$h(v,u) = \omega(v,Ju) - i\omega(v,u) = \omega(u,Jv) + i\omega(u,v) = \overline{h(u,v)}$$

In other words, h is Hermitian, as desired. □

In coordinates, fixing a basis $z_1, .., z_n$ of V, denote $h(z_i, z_j) = h_{ij}$. Then if $u = (u_1, ..., u_n), v = (v_1, ..., v_n)$, we have $h(u,v) = \sum h_{ij} u_i \overline{v}_j$ and can thus write

$$h = \sum h_{ij} z_i^* \otimes \overline{z_j}^*$$

Now,

$$\omega(u,v) = -Im(h(u,v)) = \frac{-1}{2i}(h(u,v) - \overline{h(u,v)}) = \frac{i}{2}(h(u,v) - h(v,u))$$

i.e.

$$\omega(u,v) = \frac{i}{2}\sum h_{ij}(u_i\overline{v}_j - v_i\overline{u}_j)$$

which means we can identify

$$w = \frac{i}{2}\sum h_{ij} z_i^* \wedge \overline{z}_j^*$$

since $[z_i^* \wedge \overline{z}_j^*](u,v) = u_i\overline{v}_j - v_i\overline{u}_j$.

2.8.2 Hermitian forms and Kähler forms on complex manifolds

Now, on a general complex manifold, we always need to bear in mind the identification $T^{\mathbb{R}}X \simeq T^{1,0}X$ given by $v \mapsto \frac{1}{2}(v - iJv)$. We see that there are three types of equivalent structures on the linearization: the J-invariant Riemannian metrics g on the real tangent bundle, the Hermitian metrics on the holomorphic tangent spaces and the real $(1,1)$-forms. Moreover, any 2 of these determine the third via the equality

$$h = g - i\omega$$

Definition 2.52 (Hermitian metric and Kähler forms): *A Hermitian metric on a complex manifold X is a collection h_x of Hermitian forms on the holomorphic tangent space $T_x^{1,0}X$. To such a collection of metrics we can associate the real 2-form of type $\omega = -Im(h) \in \Omega^2_{X,\mathbb{R}} \cap \Omega^{1,1}_X$, called the fundamental form. This form is called Kähler if it is closed, and this is equivalent to the matrix h_{ij} defining h being positive definite, by the fact that $d\omega = 0$ is equivalent to the osculating of h to the standard metric in some coordinate system. Finally, we can also associate $g = \mathfrak{Re}(h)$, the associated Riemannian metric, again identifying the holomorphic and real tangent spaces. The inverse identification is given by $g = \omega(-, J-)$.*

Remark: In the lectures, we divided by 2 - this comes down to the fact that we put the half when we identified the real and holo tangent bundles, i.e. we put $v \mapsto \frac{1}{2}(v - iJv)$.

The canonical example of a Kähler form is the 2-form associated to the standard inner product on $\mathbb{C}^n$, namely the form $\omega = \frac{i}{2}\sum dz_i \wedge d\overline{z}_i$.

The fundamental form is related to the volume form of a complex manifold. Given a complex manifold X with a Hermitian metric h, let $e_1, ..., e_n \in T_xX$ be an orthonormal basis for h_x over $\mathbb{C}$. Then $e_1, Je_1, ..., e_n, Je_n$ is orthonormal for $g_x = Re(h_x)$, the Riemannian metric. Denote the dual basis for $\Omega_{X,x,\mathbb{R}}$ to be $dx_1, dy_1, ..., dx_n, dy_n$ and put $dz_i = dx_i + idy_i$. We have

$$w_x = \frac{i}{2}\sum dz_i \wedge d\overline{z}_i$$

and hence

$$\frac{w_x^n}{n!} = (\frac{i}{2})^n dz_1 \wedge d\overline{z}_1 ... \wedge dz_n \wedge d\overline{z}_n = dx_1 \wedge dy_1 ... \wedge dx_n \wedge dy_n$$

and so $w^n/n!$ is a volume form. In particular,

$$\mathrm{Vol}(X) = \int_X \frac{w^n}{n!} > 0$$

If ω is Kähler, then ω^k is closed for all k, so it defines a nontrivial De Rham cohomology class in $H^{2k}(X,\mathbb{R})$ for the following reason: if $\omega^k = d\eta$, then $\omega^n = d(\omega^{n-k} \wedge \eta)$ and by Stokes' theorem we will get that $\mathrm{Vol}(X) = 0$, which is impossible. Hence, if a manifold is Kähler, it has to have nontrivial de Rham cohomology groups in all even dimensions.

2.8.3 Projective space and the Fubini-Study metric

In this section, we show that any projective manifold is Kähler, using the Fubini-Study metric[3].

Recall that complex projective space $\mathbb{P}^n$ has an open cover consisting of $U_i = \{(w_0 : ... : w_n)|z_i \neq 0\} \simeq \mathbb{C}^n$ sending $(w_0 : ... : w_n) \mapsto (\frac{w_0}{w_i}, ..., \frac{w_n}{w_i})$ (omitting the i-th component which is just 1). On this patch, define

$$\omega_i = \frac{i}{2\pi}\partial\overline{\partial}\log(\sum |\frac{w_j}{w_i}|^2)$$

Under the trivialization $\phi_i : U_i \to \mathbb{C}^n$, this corresponds to

$$\frac{i}{2\pi}\partial\overline{\partial}\log(1 + \sum |z_k|^2)$$

We want to show that these glue to a global closed form on $\mathbb{P}^n$. But

$$\log(\sum |\frac{w_j}{w_i}|^2) = \log(|\frac{w_k}{w_i}|^2 \sum |\frac{w_j}{w_k}|^2)) = \log(|\frac{w_k}{w_i}|^2) + \log(\sum |\frac{w_j}{w_k}|^2)$$

So to show ω_i and ω_k agree on $U_k \cap U_i$, we need to show that

$$\partial\overline{\partial}\log(|\frac{w_k}{w_i}|^2) = 0$$

When $i < k$, on U_i the function w_k/w_i corresponds to the $k-$th coordinate on $\mathbb{C}^n$ under ϕ_i. But

$$\partial\overline{\partial}\log(|z|^2) = \partial(\frac{1}{z\bar{z}}\overline{\partial}(z\bar{z})) = \partial(\frac{zd\bar{z}}{z\bar{z}}) = \partial(\frac{d\bar{z}}{\bar{z}}) = 0$$

Hence the ω_i glue together to a global form ω. Moreover, it is clear that $d\omega = \partial\omega = \overline{\partial}\omega = 0$, using the fact that $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$, so ω is closed. Moreover, $\overline{w_i} = w_i$ using the fact that $\overline{\partial\overline{\partial}} = \overline{\partial}\partial = -\partial\overline{\partial}$, hence ω is real.

To show that ω arises from a metric, we need to show that its matrix is positive semidefinite. However,

$$\partial\overline{\partial}(1 + \sum |z_k|^2) = \frac{\sum dz_i d\bar{z}_i}{1 + \sum |z_i|^2} - \frac{(\sum \bar{z}_i dz_i) \wedge (\sum z_i d\bar{z}_i)}{(1 + \sum |z_i|^2)^2} = \frac{1}{(1 + \sum |z_i|^2)^2} \sum h_{ij} dz_i d\bar{z}_j$$

[3] Note that the Fubini-Study metric can also be defined as the Chern form associated to the tautological line bundle over projective space - see Voisin 3.3.1

Now $h_{ij} = (1 + \sum |z_i|^2)\delta_{ij} - \bar{z}_i z_j$ which is positive by using the Cauchy-Schwarz inequality:

$$x^t(h_{ij})\bar{x} = x^t I \bar{x} + x^t z^t \bar{z} x - x^t \bar{z} z^t \bar{x} = (x,x) + (x,x)(z,z) - |(x,z)|^2 > 0$$

This completes the demonstration that projective space admits a Kähler metric.

Remark: The $\partial\overline{\partial}$ is not coincidental: after showing the Hodge decomposition for Kähler manifolds, one can prove the so-called $\partial\overline{\partial}$-lemma, which says that any d-closed form is locally $\partial\overline{\partial}$-exact.

Remark: Any complex submanifold of a Kähler manifold inherits the Hermitian metric and moreover inherits a Kähler form, and hence is Kähler. We have thus shown that any projective manifold is Kähler.

2.9 Hodge theory of Kähler manifolds

We conclude with a section proving the Hodge decomposition for compact Kähler manifolds. To do this, we first define the Hodge star operator, which will allow us to construct duals, or adjoints, of the operators d, ∂ and $\overline{\partial}$. We then define Laplacians and harmonic forms, and show that any cohomology class can be represented uniquely by an element of the vector space of harmonic forms. Then, we prove the Kähler identities, which allow us to show that the harmonic k-forms split into a sum of the harmonic (p,q)–forms, and then we conclude by using the isomorphism between k-th cohomology and the k-th harmonic forms.

2.9.1 The Hodge star and adjoints on smooth manifolds

Let X be a compact smooth manifold with a Riemannian metric. This induces a metric on the differential forms as follows: if $e_1, ..., e_n$ is an orthonormal basis for $T_{X,x}$, then the $e_{i_1}^* \wedge ... \wedge e_{i_k}^*$ form an orthonormal basis for $\Omega_{X,x}^k$.

Definition 2.53 (Hodge star): *The Hodge $*$ operator is the unique operator $\Omega_X^k \to \Omega_X^{n-k}$ such that*

$$\alpha \wedge *\beta = \{\alpha, \beta\}\mathrm{Vol}$$

where $\alpha, \beta \in \mathcal{A}^k(X)$ are sections of Ω_X^k and $*$ is induced from the operator on bundles given by composing a section with $*$.

The existence of the Hodge star operator is guaranteed by the following reasoning:

Firstly, we have the isomorphism

$$\Omega_{X,x}^{n-k} \simeq \mathrm{Hom}(\Omega_{X,x}^k, \Omega_{X,x}^n)$$

given by the right wedge product. This is an isomorphism, as the map is clearly injective and also the two vector spaces have the same dimension. Note that when the manifold is Riemannian, it has a volume form so $\Omega^n_{X,x}$ is canonically isomorphic to $\mathbb{R}$. Moreover, the metric gives us an isomorphism

$$\Omega^k_{X,x} \simeq \mathrm{Hom}(\Omega^k_{X,x}, \mathbb{R})$$

given by $\omega \mapsto (-,\omega)$. Composing these isomorphism we have:

$$\Omega^k_{X,x} \simeq \mathrm{Hom}(\Omega^k_{X,x}, \mathbb{R}) \simeq \mathrm{Hom}(\Omega^k_{X,x}, \Omega^n_{X,x}) \simeq \Omega^{n-k}_{X,x}$$

Denoting this composite map $*$ and unraveling the definitions, we see that for a section $\beta \in \mathcal{A}^k(X)$, $*\beta$ is the element in $\mathcal{A}^{n-k}(X)$ such that wedging with it produces the same map as using the metric:

$$- \wedge *\beta = \{-, \beta\}\mathrm{Vol}$$

.

Definition 2.54 (L^2 metric): *On elements $\alpha, \beta \in \mathcal{A}^k(X)$ we have the L^2 metric defined by*

$$(\alpha, \beta)_{L^2} = \int_X \{\alpha, \beta\}\mathrm{Vol},$$

where $x \mapsto \{\alpha_x, \beta_x\}_x$ is a function of x.

Immediately from the definition we see that $(\alpha, \beta)_{L^2} = \int_X \alpha \wedge *\beta$.

Proposition 2.55: *The Hodge star operator satisfies $*^2 = (-1)^{k(n-k)}$.*

Proof. $*$ preserves metrics, so we have

$$\alpha_x \wedge *\beta_x = (\alpha_x, \beta_x)_x \mathrm{Vol}_x = (*\alpha_x, *\beta_x)_x \mathrm{Vol}_x = *\beta_x \wedge ** \alpha_x = (-1)^{k(n-k)} ** \alpha_x \wedge *\beta_x$$

□

Let $d : \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)$ be the exterior derivative and define $d^* = (-1)^k *^{-1} \circ d \circ * = (-1)^{n(k+1)+1} * \circ d \circ *$. This is called the adjoint to d for the following reason:

Proposition 2.56 (Adjoint property): *If X is compact or only compactly supported integration is allowed, then*

$$(\alpha, d^*\beta)_{L^2} = (d\alpha, \beta)_{L^2}$$

Proof. Let $\alpha \in \mathcal{A}^{k-1}(X), \beta \in \mathcal{A}^k(X)$. Then $(d\alpha, \beta)_{L^2} = \int_X d\alpha \wedge *\beta$. However, $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{k-1}\alpha \wedge d * \beta$. Integrating over X and using Stokes' theorem, we get

$$(d\alpha, \beta)_{L^2} = (-1)^k \int_X \alpha \wedge d * \beta$$

But $(\alpha, d^*\beta)_{L^2} = \int_X \alpha \wedge *(-1)^k *^{-1} d * \beta = (-1)^k \int_X \alpha \wedge d * \beta$ and so the two quantities are equal. □

Remark (Local expression for the Hodge star and the adjoint): Locally, for a form $u = u_K dz_J$ we have that

$$d^* u = -\sum_{l=1}^{k} \frac{\partial u_J}{\partial x_l} \iota_{\partial x_l} dx_J$$

2.9.2 The operators ∂ and $\overline{\partial}$ on complex manifolds

The Hodge star operator was defined for smooth manifolds in the previous section. Now let X be a compact complex manifold. We can extend the Riemannian metric to a Hermitian metric on the complexified cotangent bundle and extend $*$ $\mathbb{C}$-linearly to complex-valued forms. In local coordinates, if

$$\alpha = \sum \alpha_{I,J} dz_I \wedge d\overline{z}_J, \beta = \sum \beta_{I,J} dz_I \wedge d\overline{z}_J$$

are in $\Omega_X^{p,q}$, then their Hermitian product at x is equal to

$$(\alpha_x, \beta_x)_x = \sum \alpha_{I,J}(x)\overline{\beta_{I,J}(x)}$$

We define the complex Hodge star by the identity

$$\{\alpha_x, \beta_x\}_h \mathrm{Vol}_x = \alpha_x \wedge \overline{*\beta_x}$$

and the Hodge star takes a (p,q) form to an $(n-q, n-p)$ form - note the swapping of p and q! It gives an isometry

$$\Omega^{p,q} \to \Omega^{n-q.n-p}$$

Recall that $d = \partial + \overline{\partial}$ on complex manifolds and since a complex manifold has underlying even dimension, then $(-1)^{n(k+1)+1} = -1$ and we can define the duals of ∂ and $\overline{\partial}$ to be $\partial^* = -*\overline{\partial}*, \overline{\partial}^* = -*\partial*$. These satisfy the same adjoint property as d:

Proposition 2.57: *∂^* and $\overline{\partial}^*$ are formal adjoints of ∂ and ∂^* respectively.*

$$\begin{tikzcd} \Gamma(\Omega_X^{p,q}) \arrow[r, bend left, "\partial"] & \Gamma(\Omega_X^{p+1,q}) \arrow[l, bend left, "\partial^*"] \end{tikzcd} \qquad \begin{tikzcd} \Gamma(\Omega_X^{p,q}) \arrow[r, bend left, "\overline{\partial}"] & \Gamma(\Omega_X^{p,q+1}) \arrow[l, bend left, "\overline{\partial}^*"] \end{tikzcd}$$

Proof. We show this for ∂, the other case being analogous. Firstly, if α is of type $(p-1,q)$ and β of type (p,q) with $n = p+q$ beng the dimension of X as a complex manifold, then

$$(\partial\alpha, \beta)_{L^2} = \int_X \partial\alpha \wedge \overline{*\beta}$$

However, $\alpha \wedge \overline{*\beta}$ is of type $(n-1,n)$ and hence $\partial(\alpha \wedge \overline{*\beta}) = d(\alpha \wedge \overline{*\beta})$. Now by Stokes' theorem,

$$0 = \int_X d(\alpha \wedge \overline{*\beta}) = \int_X \partial(\alpha \wedge \overline{*\beta}) = (\partial\alpha, \beta)_{L^2} + (-1)^{p+q-1} \int_X \alpha \wedge \partial\overline{*\beta} \tag{2.1}$$

But note that

$$(\alpha, \partial^* \beta)_{L^2} = \int_X \alpha \wedge * \overline{\partial^* \beta} = \int_X \alpha \wedge \overline{* - * \overline{\partial} * \beta} = (-1) \int_X \alpha \wedge * * \partial * \overline{\beta}$$

The last equality comes from the fact that $*$ is a real operator. But $\partial * \overline{\beta}$ is a form of type $(n-p+1, n-q)$ on which $**$ acts as $(-1)^{2n-p-q+1} = (-1)^{p+q+1}$ and hence

$$(\alpha, \partial^* \beta)_{L^2} = (-1)^{p+q} \int_X \alpha \wedge \partial * \overline{\beta} \tag{2.2}$$

Combining (6.1) and (6.2) gives the result. □

Remark: The preceding constructions can also be extended to the case of holomorphic vector bundles with the operator $\overline{\partial}_E$ as in the Dolbeault complex of a holomorphic vector bundle.

Remark (Local expressions): Locally, for $v = v_{J,K} dz_J \wedge d\overline{z}_K$ we have

$$*v = \epsilon_{J,K} v_{J,K} dz_{CJ} \wedge d\overline{z}_{CK}$$

where C denotes the complement and $\epsilon_{J,K}$ is the sign of the permutation $(-1)^{q(n-p)} \mathrm{sign}(J, CJ) \mathrm{sign}(K, CK)$. Using this, we can find a local expression

$$\overline{\partial}^* v = -* \partial * v = -\sum_{l=1}^{n} \frac{\partial v_{J,K}}{\partial z_l} \iota_{\overline{\partial z_l}} dz_J \wedge d\overline{z}_K$$

In other words, we are differentiating the functions and contracting the form part with the conjugate.

2.9.3 Laplacians, harmonic forms and cohomology

For any differential operator θ, e.g. d, ∂ or $\overline{\partial}$ define its associated Laplacian as

$$\Delta_\theta = \theta\theta^* + \theta^*\theta$$

As a corollary of the adjunction properties 2.56 and 2.57 we have:

Corollary 2.58: $(\alpha, \Delta_\theta \beta)_{L^2} = (\theta\alpha, \theta\beta)_{L^2} + (\theta^*\alpha, \theta^*\beta)_{L^2}$. *In particular,* $(\alpha, \Delta_\theta \alpha)_{L^2} = \|\theta\alpha\|^2 + \|\theta^*\alpha\|^2$.

Definition 2.59 (Harmonic forms): *A θ-harmonic form is a form α such that* $\Delta_\theta \alpha = 0$

Hence, by applying 2.58, we see that a form is θ-harmonic if and only if it is θ and θ^*-closed:

Corollary 2.60: $\ker \Delta_\theta = \ker \theta \cap \ker \theta^*$

Definition 2.61 (Vector space of harmonic forms): *Define $\mathcal{H}_d^k$ (resp. $\mathcal{H}_{\overline{\partial}}^k$) to be the space of all d (resp. $\overline{\partial}$)-harmonic forms, and $\mathcal{H}_d^{p,q}$ (resp. $\mathcal{H}_{\overline{\partial}}^{p,q}$) the d (resp. $\overline{\partial}$)-harmonic forms of type (p,q).*

Now we show that the De Rham cohomology groups are isomorphic to these harmonic vector spaces, using a big theorem about elliptic differential operators which we quote without proof:

Theorem 2.62 (Big theorem on elliptic differential operators): *Let $P : E \to F$ be an EDO on a compact manifold. If E and F are of the same rank and are equipped with metrics, then $\ker P$ is of finite dimension and there is an L^2 orthogonal decomposition*

$$C^\infty(E) = \ker P \oplus P^*(C^\infty(F),$$

where P^* is the formal adjoint of P.

We will apply this to the Laplacian Δ_d, which is an elliptic differential operator of degree 2, which is also self-adjoint: $\Delta = \Delta^*$. In particular, we have

$$\mathcal{A}^k(X) = \mathcal{H}^k \oplus \Delta(\mathcal{A}^k(X))$$

Now let's see what happens when we pass to cohomology: let β be a closed form, $\beta = \alpha + \Delta\gamma$ with α harmonic, i.e. $\beta = \alpha + dd^*\gamma + d^*d\gamma$. But now β, α and $dd^*\gamma$ are all closed, hence $d^*d\gamma$ is closed, $d^*d\gamma \in \ker d \cap \operatorname{im} d^*$. But $0 \leq (d^*d\gamma, d^*d\gamma) = (d\gamma, dd^*d\gamma) = 0$ and hence $d^*d\gamma = 0$. Hence β is represented by a harmonic form modulo some exact form, and the map $\mathcal{H}^k \to H^k(X)$ is surjective. Conversely, to show injectivity, assume β is harmonic and exact. Then $\beta \in \ker d^* \cap \operatorname{im} d$ and again it must be 0. We conclude that:

Theorem 2.63: *Let X be a compact oriented Riemannian manifold. Then the map*

$$\mathcal{H}^k \to H^k(X)$$

is an isomorphism

Remark: Note that this statement applies to the real-valued De Rham cohomology when dealing with the usual exterior derivative, but also works when we extend it $\mathbb{C}$-linearly with complex-valued De Rham cohomology.

We can apply the same idea to the Laplacian associated to $\overline{\partial}$ to get the decomposition

$$C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,q}))$$

Using the exact same reasoning, we get the following:

Theorem 2.64: *Let X be a compact complex manifold with a Hermitian metric. Then the map*

$$\mathcal{H}^{p,q} \to H^{p,q}(X)$$

is an isomorphism. In particular, the Dolbeault cohomology groups have finite dimension.

2.9.4 The case of Kähler manifolds

Our aim now is to use the isomorphism between the harmonic and ordinary cohomology groups, together with the decomposition $\mathcal{H}^k = \bigoplus \mathcal{H}^{p,q}$ to prove the Hodge decomposition theorem, the final theorem in this note.

To do this, we will work entirely with compact Kähler manifolds (the decomposition theorem does not necessarily hold for non-Kähler manifolds) and prove the so-called Kähler identities to establish the equality between the different Laplacians acting on X.

Definition 2.65 (Lefschetz operator): *Define the Lefschetz operator on complex differential forms*

$$L : \mathcal{A}_X^k \to \mathcal{A}_X^{k+2}$$

by $\alpha \mapsto \omega \wedge \alpha$, where ω is the Kähler form. Its formal dual is

$$\Lambda : \mathcal{A}_X^k \to \mathcal{A}_X^{k-2}$$

where $\Lambda = (-1)^k * L *$

The construction of the adjoint can be verified by seeing that

$$\alpha \wedge *\Lambda\beta = (\alpha, \Lambda\beta)\mathrm{Vol} = (L\alpha, \beta)\mathrm{Vol} = L\alpha \wedge *\beta = \alpha \wedge \omega \wedge *\beta$$

i.e. $*\Lambda = L*$, or $\Lambda = *^{-1}L*$.

Lemma 2.66 (Local Kähler identity): *Let $U \subset \mathbb{C}^n$ be a region with the constant metric $\omega = i\sum dz_i \wedge d\bar{z}_i$. Then we have*

$$[\overline{\partial}^*, L] = i\partial$$

Proof. Let's use 2.9.2:

$$\overline{\partial}^* v = - * \partial * v = -\sum_{l=1}^{n} \frac{\partial v_{J,K}}{\partial z_l} \iota_{\overline{\partial z_l}} dz_J \wedge d\bar{z}_K$$

We can now compute and use that contraction is a derivation.

$$[\overline{\partial}^*, L]v = \overline{\partial}^*(v \wedge \omega) - \overline{\partial}^* v \wedge \omega = i\left[\overline{\partial}^*(v_{J,K} dz_J \wedge d\overline{z}_K \wedge dz_i \wedge d\overline{z}_i) + \frac{\partial v_{J,K}}{\partial z_l}(\overline{\partial}_l \lrcorner dz_J \wedge d\overline{z}_K) dz_i \wedge d\overline{z}_i\right] =$$

$$= i\left[-\frac{\partial v_{J,K}}{\partial z_l}\overline{\partial}_l \lrcorner (dz_J \wedge d\overline{z}_K \wedge dz_i \wedge d\overline{z}_i) + \frac{\partial v_{J,K}}{\partial z_l}(\overline{\partial}_l \lrcorner dz_J \wedge d\overline{z}_K) dz_i \wedge d\overline{z}_i\right] =$$

$$= i\left[(-1)^{p+q+1}\frac{\partial v_{J,K}}{\partial z_l} dz_J \wedge d\overline{z}_K \overline{\partial}_l \lrcorner (dz_i \wedge d\overline{z}_i)\right] = i\left[(-1)^{p+q+2}\frac{\partial v_{J,K}}{\partial z_l} dz_J \wedge d\overline{z}_K \wedge dz_l\right] =$$

$$= i\frac{\partial v_{J,K}}{\partial z_l} dz_l \wedge dz_J \wedge d\overline{z}_K = i\partial v$$

□

Proposition 2.67 (Kähler identities):

$$[\overline{\partial}^*, L] = i\partial, [\partial^*, L] = -i\overline{\partial}$$

$$[\Lambda, \overline{\partial}] = -i\partial^*, [\Lambda, \partial] = i\overline{\partial}^*$$

Proof. Since ω is real, the Lefschetz operator and its adjoint are as well, i.e. $L = \overline{L}$ and hence the identities come in conjugate pairs. Also, the first one on the first row implies the first one on the second row by adjointness:

$$([\Lambda, \overline{\partial}]u, v)_{L^2} = (u, [\overline{\partial}^*, L]v)_{L^2} = (u, i\partial v)_{L^2} = (-i\partial^* u, v)_{L^2}$$

However, the first one follows by the fact that the Kähler metric is the standard metric up to order two in a nice set of coordinates, in which case we just invoke the local computation from 2.66, which only uses terms up to order 1. □

Corollary 2.68 (Comparing the Laplacians): *We have* $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$

Proof.

$$\Delta_d = dd^* + d^*d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \tag{2.3}$$

Notice that by the Kähler identities,

$$\partial^*\overline{\partial} = i[\Lambda, \overline{\partial}]\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial}$$

and similarly

$$\overline{\partial}\partial^* = i\overline{\partial}\Lambda\overline{\partial}$$

i.e.

$$\partial^*\overline{\partial} = -\overline{\partial}\partial^*$$

Also, note that we have $\partial\overline{\partial} = -\overline{\partial}\partial$.

Expanding (6.3) we get

$$\Delta_d = \partial\partial^* + \partial\overline{\partial}^* + \overline{\partial}\partial^* + \overline{\partial}\overline{\partial}^* + \partial^*\partial + \partial^*\overline{\partial} + \overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial}$$

Now, the gray bits are both 0 and we are left with

$$\Delta_d = \Delta_\partial + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

But $\overline{\partial}^* = -i[\Lambda, \partial]$ so we get

$$\Delta_d = \Delta_\partial + \overline{\partial}(-i\Lambda\partial + i\partial\Lambda) + (-i\Lambda\partial + i\partial\Lambda)\overline{\partial} = \Delta_\partial + i\partial[\Lambda, \overline{\partial}] + i[\Lambda, \overline{\partial}]\partial = 2\Delta_\partial$$

The other case is proved in exactly the same way. □

Now, since Δ_∂ is bihomogenous, i.e. keeps the bigrading the same, the same will apply to Δ_d. Hence, if we have a d-harmonic form $\alpha = \sum \alpha^{p,q}$, we deduce that each $\alpha^{p,q}$ is d-harmonic. In other words,

Theorem 2.69:

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

Notice that $\mathcal{H}^{p,q} = \overline{\mathcal{H}^{q,p}}$ since if β is harmonic of type (p,q), then $\overline{\beta}$ is of type (q,p) and

$$\overline{\Delta_\partial \overline{\beta}} = \Delta_{\overline{\partial}}\beta = \Delta_\partial \beta = 0$$

i.e. $\overline{\beta}$ is harmonic as well.

Now recall that by theorem 2.63 we have that

$$\mathcal{H}^{p,q} \simeq H^{p,q}(X)$$

and

$$\mathcal{H}^k \simeq H^k(X)$$

This allows us to conclude:

Theorem 2.70 (Hodge decomposition): *We have the decomposition*

$$H^k(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

Example (Hodge numbers of projective space): We have that $H^{p,p}(\mathbb{CP}^n) \simeq \mathbb{C}\omega^p, H^{p,q}(\mathbb{CP}^n) = 0$.

Corollary 2.71 (Holomorphic forms are closed): *Given $\alpha \in H^0(X, \Omega^p) = H^{p,0}(X)$, i.e. $\overline{\alpha} = 0$, we also have that, for degree reasons, $\overline{\partial}^* \alpha = 0$ and hence $\Delta_{\overline{\partial}}\alpha = 0$ and hence $\Delta_d \alpha = 0$ and so $d\alpha = 0$.*

Corollary 2.72 (Del del bar lemma): *Suppose v is a d-closed, $\overline{\partial}$–exact (p,q)–form. Then it is of the form $\partial\overline{\partial}\varphi$*

Proof. Put $\upsilon = \overline{\partial}\eta$. We have, by the decomposition, $\eta = \alpha + \partial\beta + \partial^*\gamma$ with α harmonic for each of the three operators, since we are on a Kähler manifold. We get

$$\upsilon = \overline{\partial}\partial\beta + \overline{\partial}\partial^*\gamma = -\partial\overline{\partial}\beta - \partial^*\overline{\partial}\gamma$$

But now since $\partial\upsilon = 0$, we must have that $\partial\partial^*\overline{\partial}\gamma = 0$, i.e. $\overline{\partial}\gamma \in \ker(\partial) \cap \operatorname{im}\partial^* = 0$. This follows, since if $\partial^* x$ is in the kernel of ∂, then $0 = (x, \partial\partial^* x) = \|\partial^* x\|^2$ so $\partial^* x = 0$. $\square$

3 Algebraic Geometry

3.1 Affine algebraic geometry basics

An affine variety is a zero locus $V = \mathbb{V}(S)$ where $S \subset k[x_1, \dots, x_n]$ is some set of polynomials.

3.1.1 The Zariski topology

Definition 3.1 (Zariski topology): *The topology on an affine variety can be defined either as the one induced by the Zariski topology on $\mathbb{A}^n$ or by using coordinate rings: $W \subset V$ is closed iff $W = \mathbb{V}(\overline{J})$ with $\overline{J}$ an ideal in $k[V] = k[x_1, \dots, x_n]/I(S)$.*

These turn out to be equivalent definitions: suppose W is closed in the intrinsic topology. Then the preimage of $\overline{J}$ via $k[x_1, \dots, x_n] \xrightarrow{\pi} k[V]$ is an ideal J which contains $I = I(S)$. Hence $W = \mathbb{V}(J) = \mathbb{V}(J) \cap V$ is closed in the induced Zariski topology, i.e. the extrinsic topology - this is because $I \subseteq J$ implies $\mathbb{V}(J) \subseteq \mathbb{V}(I)$.

Conversely, if $W = \mathbb{V}(J) \cap V = \mathbb{V}(J) \cap \mathbb{V}(I) = \mathbb{V}(I + J)$ this immediately means that it is of the form $W = \mathbb{V}(\overline{I + J})$.

3.1.2 The Nullstellensatz

To any affine variety, using the Nullstellensatz, we can first associate its coordinate ring and then look at $MaxSpec(k[V])$ with the Zariski topology - this recovers the points of V! To prove this, one basically sees that any point in V naturally gives a maximal ideal (the kernel of the evaluation map). Conversely, a maximal ideal $\mathfrak{m} \subset k[x_1, \dots, x_n]$ gives us a field extension $k \subseteq k[\underline{x}]/\mathfrak{m} = K$ - this is a finite algebraic extension by prop. 7.9 in Atyiah-Macdonald. But k is algebraically closed, hence $k = K$, and the images of x_i must be in k, i.e. $\mathfrak{m} \subseteq (x_1 - a_1, \dots, x_n - a_n)$ and this must be an equality by maximality of $\mathfrak{m}$.

From this we deduce that since any ideal is contained in a maximal ideal, by taking $\mathbb{V}$, any nonunit ideal contains some $\mathbb{V}(\mathfrak{m})$, which is a point, i.e. is nonempty. This also allows us to prove the strong Nullstellensatz, by using the Rabinowitsch trick: consider an affine variety V defined by an ideal I. Then, for some $f \in I(V)$, consider the ideal of $k[x_1, \dots, x_n, T]$ generated by the image of I and also $1 - fT$ and call this J. We can see that the zero set of this is empty, since if all things in I vanish on some point a, then $a \in V$ and hence $f(a) = 0$, contradicting the fact that we need $1 - fT$ to vanish as well. Therefore $\mathbb{V}(J) = \emptyset$ which by the weak Nullstellensatz implies that J is the unit ideal, i.e. $1 = \sum T^r h_r + (1 - fT)g$ where $h_r \in I$ and $g \in k[x_1, \dots, x_n, T]$. Setting $T = 1/f$ then shows that some power of f is in I, i.e. $I(\mathbb{V}(I)) \subseteq \sqrt{I}$. The other inclusion is easy.

Another way to see this is to apply the Zariski lemma again: take an f that is not in $\sqrt{I}$. In other words, it is not in the intersection of all primes containing I and hence there must be some prime $\mathfrak{p}$ containing I but not f. Form the composition

$$k[x_1,\dots,x_n] = R \to R/\mathfrak{p} \to (R/\mathfrak{p})[f^{-1}] \to (R/\mathfrak{p})[f^{-1}]/\mathfrak{m} \simeq k$$

$\mathfrak{m}$ is a maximal ideal of the localization ring, and the quotient by it is a finite extension of k, which must be k itself since $\overline{k} = k$. Since $I \subseteq \mathfrak{p}$ we have that the image of the coordinates x_i produce a point $(a_1,\dots,a_n) \in k^n$ which is in $\mathbb{V}(I)$ and also we must have $f(a_1,\dots,a_n) \neq 0$. This means that if f is not in the radical of I, then it is not in $\mathbb{V}(I)$.

Schemes are introduced to take into account double points - for instance, when we look at the intersection of $y - x^2$ and y, this gives a single point using the maximal ideal interpretation, but really the coordinate ring is $k[x]/(x^2)$ - this has only one maximal ideal but two prime ideals!

3.1.3 The main correspondence

Proposition 3.2 (Main correspondence): *Affine varieties correspond precisely to the finitely generated reduced k-algebras, by taking their coordinate rings.*

$$\{\textit{Affine algebraic varieties}\} \leftrightarrow \{\textit{F.g. reduced k-algebras}\}$$

In fact, this is an equivalence of categories, where one has to be taken with opposite arrows.

On one hand, the association is easy: given a map $V \to W$ between affine varieties, we can just form the pullback $k[W] \to k[V]$. The association on the other side works as follows: given a map $\varphi : k[X]/I \to k[Y]/J$, where $X = \{x_1,\dots,x_n\} Y = \{y_1,\dots,y_m\}$ we first create $f_i = \varphi(\overline{x_i})$ and pick representatives of f_i in $k[Y]$. Now we want to produce a map $\mathbb{V}(J) \to \mathbb{V}(I)$. We do this by sending $a \in \mathbb{V}(J)$ to $(f_1(a),\dots,f_n(a)) \in k^n$. The choice of representatives of f_i doesn't matter since we are in $\mathbb{V}(J)$ i.e. two different choices will have a difference that vanishes on it and we get the same result. We have to show that the image is contained in $\mathbb{V}(I)$.

To see this, let $g \in I$. Then $g(f_1(a),\dots,f_n(a)) = \varphi(\overline{g(x_1,...,x_n)})(a) = 0$ (we use the fact that φ is a k-algebra homomorphism). More precisely, we have the diagram:

$$\begin{tikzcd} k[X] \arrow[d] & k[Y] \arrow[d] & \\ k[X]/I \arrow[r, "\varphi"] & k[Y]/J \arrow[r, "\overline{ev_a}"] & k \end{tikzcd}$$

For any $a \in \mathbb{V}(J)$, we have that $J \subseteq \mathfrak{m}_a$ and hence we get an induced evaluation map $k[Y]/J \to k$. What we are really doing is taking the images of x_i in k and bunching them up into one element of k^n, i.e. to $a \in \mathbb{V}(J)$ we associate an element $\omega = (\overline{ev_a} \circ \varphi(\overline{x_1}),\dots,\overline{ev_a} \circ \varphi(\overline{x_n})) \in k^n$. To see this is in $\mathbb{V}(I)$ we check $g(\omega) = \overline{ev_a} \circ \varphi(\overline{g(x_1,\dots,x_n)}) = 0$ when $g \in I$.

3.2 Schemes

3.2.1 Affine Schemes

The moral of the whole story is that coordinate rings are nice and are more of an intrinsic invariant of an affine variety, i.e. independent of how it is embedded in affine space. However, the association $V \mapsto \mathrm{MaxSpec}k[V]$ is not functorial - what we actually need is Spec! As already mentioned, this takes care of double points and is functorial, so is much better.

Definition 3.3 (Affine schemes): *An affine scheme is a topological space of the form* $\mathrm{Spec}(A)$. *We should think of A as the ring of global functions on this space. In particular, evaluating $a \in A$ at a prime ideal $\mathfrak{p}$ is done via the sequence of maps $A \to A/\mathfrak{p} \to \kappa(\mathfrak{p})$. we then define the closed sets of the Zariski topology as "vanishing sets":*

$$\mathbf{V}(f) = \{\mathfrak{p} | ev_{\mathfrak{p}}(f) = 0\} = \{\mathfrak{p} | f \in \mathfrak{p}\}$$

In general, $\mathbf{V}(I) = \{\mathfrak{p} | I \subseteq \mathfrak{p}\} \simeq \mathrm{Spec}(A/I)$. *Also, the basic open sets are the complements of* $\mathbf{V}(f)$*:*

$$\mathbf{D}(f) = \{\mathfrak{p} | f \notin \mathfrak{p}\} \simeq \mathrm{Spec}(A[1/f])$$

In this way, closed subsets of affine schemes are naturally affine schemes, and also basic opens have the structure of affine schemes. This will allow us to define a sheaf on this topological space. Note that $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J)$ precisely because we are dealing with prime ideals, i.e. $IJ \subset I \cap J \subset \mathfrak{p}$ implies that $\mathfrak{p}$ contains one of I or J, and vice versa.

Remark: maximal ideals are closed in this topology, since $\mathbb{V}(\mathfrak{m}) = \{\mathfrak{m}\}$. Also, the generic point (0)) is dense: its closure is the whole space. **Remark**: An element $f \in A$ is nilpotent precisely when $\mathbf{D}(f)$ is empty, as in that case $f \in \cap \mathfrak{p}$, the nilradical.

To prove that $\mathbf{D}(f) \simeq \mathrm{Spec}(A[1/f])$, we need to recall some results (from Atyiah-Macdonald).

Firstly, localization is exact, and moreover the map $A \to S^{-1}A$ has the property that any ideal on the right is an extended ideal. In particular, we have a bijection between prime ideals of $S^{-1}A$ and prime ideals of A not meeting S: this is because if $\mathfrak{p} \subset A$, then $A/\mathfrak{p}$ is an integral domain and we have $S^{-1}A/S^{-1}\mathfrak{p} = \overline{S}^{-1}(A/\mathfrak{p})$, where $\overline{S}$ is the image of S mod $\mathfrak{p}$. The right hand side sits inside the field of fractions of the integral domain, so is an integral domain, hence $S^{-1}\mathfrak{p}$ is either prime in $S^{-1}A$ or the unit ideal, which happens precisely when $\mathfrak{p}$ meets S.

Hence we have a bijection between primes in $A[1/f]$ and primes in A not containing f given by extending and contracting respectively.

Now, given any map of rings $f : A \to B$, the map $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is continous with respect to the Zariski topology. This is because we have $(f^*)^{-1}(\mathbf{D}^A(g)) = \mathbf{D}^B(f(g))$. Moreover, $(f^*)^{-1}(\mathbf{V}^A(I)) = \mathbf{V}^B((fI)^+)$. Furthermore, the closure $\overline{f^*(\mathbf{V}^B(I))} = \mathbf{V}^A(f^{-1}I)$. To see this, note that from the previous property, we have that

$$(f^*)^{-1}(\mathbf{V}^A(f^{-1}(I))) = \mathbf{V}^B((ff^{-1}I)^+)$$

The thing on the right contains $\mathbf{V}^B(I)$ so by applying f^* we get that $\overline{f^*(\mathbf{V}^B(I))} \subset \mathbf{V}^A(f^{-1}I)$. For the other inclusion, assume that $\overline{f^*(\mathbf{V}^B(I))} \subset \mathbf{V}^A(J)$ for some ideal $J \subset A$, so $f^*(\mathbf{V}^B(I)) \subset \mathbf{V}^A(J)$. Then by just chasing definitions, this means that

$$J \subset \bigcap_{I \subset \eta} f^{-1}\eta = f^{-1}\sqrt{I} = \sqrt{f^{-1}I}$$

Hence, $\mathbf{V}^A(f^{-1}I) = \mathbf{V}^A(\sqrt{f^{-1}I}) \subset \mathbf{V}^A(J)$ and we are done.

In particular, applying this to the zero ideal in B, we get that if f is injective, then $\overline{f^*(\mathrm{Spec}(B))} = \mathrm{Spec}(A)$, so the image under f^* of Spec B is dense in Spec A (since the inverse image of 0 is 0 for an injective map).

Now let's get back to the case of localizations. Since every ideal in $S^{-1}A$ is an extended ideal, the map f^* is injective. Hence f^* gives a continous bijective map from $\mathrm{Spec}(S^{-1}A)$ onto its image in $\mathrm{Spec}(A)$. If we put the subspace topology on this image, we only need to show that the inverse map (the extension map) $e : \mathrm{im}(f^*) \to \mathrm{Spec}(S^{-1}(A))$ is continous. But $e^{-1}(\mathbf{D}(g)) = \mathrm{im}(f^*) \cap \{I | g \notin I^+\} = \mathrm{im}(f^*) \cap \mathbf{D}(g_0)$, where $g = g_0/s$. Hence the map is a homeomorphism onto its image.

In particular, $\mathrm{Spec}(A[1/f]) \to \mathrm{Spec}(A)$ is a homeomorphism onto the basic open set $\mathbf{D}(f)$.

Example (Fibers given by maps of affine schemes):
Whenever we have $A \to B$, then the fiber of $\mathfrak{p} \subset A$ in the induced map $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ can be identified with $\mathrm{Spec}(\kappa(\mathfrak{p}) \otimes_A B)$. This is the fiber product of $\mathrm{Spec}(B)$ with the immersion $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(A)$ representing the point $\mathfrak{p}$, and hence is the fiber of that point in $\mathrm{Spec}(B)$.

3.2.2 Sheaves, stalks and other constructions

We went over the definition of presheaves, sheaves, morphisms of sheaves, stalks. In the exercises, we proved that injectivity and surjectivity are stalk-local.

Definition 3.4 (Sheafification): *The universal free construction on a presheaf to give a sheaf:*
$$\mathcal{F}^{sh}(U) := \{(f_p) \in \prod_{p \in U} \mathcal{F}_p | \text{ for every, } p \in U \text{ there is some open neighbourhood } V_p \text{ and a section } s \in \mathcal{F}(U) \text{ such that } s_q = f_q, \forall q \in V_p\}$$
This both creates new sections that could have been created and deletes incoherent sections.

While the kernel of a sheaf is a sheaf, one needs to sheafify the image and cokernel to get sheaves!

Definition 3.5 (Direct and inverse image sheaves):
$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$ is a sheaf whenever $\mathcal{F}$ is .
For the inverse image, we first introduce the inverse image presheaf and then sheafify:

$$f^{-1}\mathcal{G}(U) = \varinjlim_{fU \subset V} \mathcal{G}(V)$$

Note that $f^*\mathcal{G}_p = \mathcal{G}_{f(p)}$. Also, the two are adjoint functors.

3.2.3 The structure sheaf on affine schemes

Consider the basic opens $\mathbf{D}(f)$ - we want to think of the ring of functions on these as the rings $A[1/f]$, since we are away from the vanishing set of f and A is our global ring of functions. So we associate rings to basic open sets in this way, which (almost) constitutes a sheaf of rings, which we denote by $\mathcal{O}$. Note that if $\mathbf{D}(g) \subset \mathbf{D}(f)$, then for all primes, $g \notin \mathfrak{p} \implies f \notin \mathfrak{p}$. The contrapositive says that $g \in \bigcap_{(f)\subset\mathfrak{p}} \mathfrak{p} = \sqrt{(f)}$. Hence, we must have a relation $g^n = uf$. So we can easily check that if $\mathbf{D}(g) = \mathbf{D}(f)$ which means we have two relations $g^n = uf, f^m = u'g$, then the rings $A[1/f] = A[1/g]$.

This relation also allows us to define restriction maps, by sending $1/f \mapsto u/g^n$. In other words, for any inclusion $\mathbf{D}(g) \subset \mathbf{D}(f)$ we get induced maps $\mathcal{O}(\mathbf{D}(f)) \to \mathcal{O}(\mathbf{D}(g))$. By an easy calculation, this is functorial, i.e. respects inclusions (composition of inclusions corresponds to compositon of restriction maps). Note that for a point $\mathfrak{p} \in \mathrm{Spec}(A)$, the taking the direct limit over all opens around it results in the stalk and we can calculate it:

Example (Stalks on the structure sheaf):

$$\varinjlim_{\mathfrak{p}\in\mathbf{D}} \mathcal{O}(\mathbf{D}) = \varinjlim_{f\notin\mathfrak{p}} \mathcal{O}(\mathbf{D}(f)) = \varinjlim_{f\notin\mathfrak{p}} A[1/f] = A_{\mathfrak{p}} = \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}}$$

Let $X = \mathrm{Spec}(A)$. Define a base sheaf on X as $\mathcal{O}_X(\mathbf{D}_f) = A_f$, which makes sense, as $\mathbf{D}_f = \mathrm{Spec}(A_f)$. This defines a sheaf on a base and can be extended uniquely to a sheaf on X, called the structure sheaf. For this sheaf, we have that $\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$.

Now, recall that $\mathbf{D}_f \cap \mathbf{D}_g = \mathbf{D}_{fg}$ and $\mathbf{D}_f \subset \mathbf{D}_g \iff f \in \sqrt{(g)}$, i.e. $f^n = ag$. Also note that $\mathbf{D}_f = \mathbf{D}_{f^n}$.

Given such an inclusion, we should have a restriction map in the sheaf, i.e. a map $A_g \to A_f$. But this is okay, as this is simply an "inclusion" $1/g \mapsto a/f^n$. We need a key result in order to prove this construction makes sense:

Proposition 3.6 (Quasicompactness of Spec A): *Affine schemes are quasicompact.*

Proof. If we have an open cover of Spec A by distinguished opens $\mathrm{Spec}A = \cup\mathbf{D}_{f_i}$, then for all primes $\mathfrak{p}$, there is some $f_i \notin \mathfrak{p}$. But in particular, this holds for the maximal ideals as well, hence the ideal generated by the f_i is not contained in any maximal ideal and is the full ring A. But then $1 = \sum f_j a_j$ for a finite number of f_j and we have quasicompactness. $\square$

To show that the sheaf on a basis satisfies the sheaf properties, we need to verify SB1 (0) and SB2 (gluing).

SB1) Suppose $s = 0$ on all $\mathbb{D}_{f_i}$. This means that $s = 0$ in the localization ring A_{f_i}, so we must have that $f_i^N s = 0$ in A for some sufficiently big N independent of j, since we have quasicompactness. Now replace f_i^N by f_i - this doesn't change the associated rings of the neighbourhoods. Now, $f_i s = 0$ for all i. But by quasicompactness, we have that $\sum f_i a_i = 1$, for some $a_i \in A$. This implies that $s = 0$.

SB2) This time, we have sections which agree on the intersections of a cover by distinguished opens. Let $s_i = t_i/f_i$ (again, replacing powers of f_i by f_i if necessary). Agreement means that $(f_i f_j)^N (t_i f_j - t_j f_i) = 0$ for all i, j. In other words,

$$f_i^N f_j^{N+1} t_i = f_j^N f_i^{N+1} t_j$$

Now find b_i with $\sum b_i f_i^{N+1} = 1$ and set $s = \sum f_i^N b_i t_i$. Notice that $f_j^N t_j = f_j^{N+1} s_i$ when restricting to $\mathbf{D}_{f_i}$. Hence, restricting s to $\mathbf{D}_{f_i}$ gives us

$$\sum f_j^N t_j b_j = \sum f_j^{N+1} s_i b_j = s_i$$

So we've glued the sections s_i to a global section s.

3.2.4 Examples of schemes

Example (Polynomial ring over a field): The scheme Spec $k[x,y]$ has three types of prime ideals of heights $2,1,0$ respectively. They are the the generic point (0) which we can think of as containing everything, the height one ideals generated by irreducible f, which we can think of as hypersurfaces in affine 2-space, whose closures in turn contain the closed points given by the maximal ideals $(x-\alpha, y-\beta)$. The local ring at the point $(x-\alpha, y-\beta)$ is given by all rational functions whose denominators are regular, i.e. don't have a pole at (α,β) - in other words, we've inverted anything that doesn't vanish at (α,β). Similarly, for the point (f) we have rational functions g/h where f does not divide h - this is DVR with maximal ideal generated by all those functions where $f|g$. Finally, for the generic point we just get the rational functions $k(x,y)$.
The spectra of these local rings should be thought of as zoomed in versions of the scheme at the point. Note that the Spec of a DVR has two points, one closed point given by the maximal ideal (f) and one other point which should be thought of as a curve passing through it, given by the generic point.
There is an important difference between spectra of quotients and spectra of local rings, i.e. localizations - Spec A/p is a closed affine subscheme, whereas Spec A_p is an open affine subscheme! The first looks like looking only at the algebraic set defined by p, whereas the second looks like the outside of that algebraic set, and moreover the first one makes p minimal, whereas the second makes p maximal.

Let's look at some simple affine schemes, and calculate them by thinking of them as fibrations.

Example (Integer polynomial ring): Let Spec $\mathbb{Z}[x]$ be thought of as a scheme over Spec $\mathbb{Z}$. We can think of this as a fibration and use the formula 3.2.1 to see that over a prime $(p) \in$ Spec $\mathbb{Z}$, we should get the affine scheme Spec$(\kappa(p) \otimes \mathbb{Z}[x]) =$ Spec$(\mathbb{F}_p[x])$. More directly, we can see that the primes lying over (p) are the primes which contain p and hence correspond to primes of $\mathbb{Z}[x]/p\mathbb{Z}[x]$. Each of these fibers is a closed affine subscheme which has its own generic point (p) together with other primes e.g. $(2, x^2 + x + 1)$ etc.
For the generic point, we get Spec $\mathbb{Q}[x]$ as the residue field $\kappa(0)$ is equal to $\mathbb{Z}[x]_{(0)} = \mathbb{Q}(x)$. This contains the generic point (0) as well as the irreducible rational polynomials f.
In terms of the topology, a lot more is going on. Take $x^2 + 1$ which lies over the generic point (0). Its closure consists of all the ideals which contain it, which for example includes $(2, x+1)$ as $x^2 + 1 = (x+1)^2 - 2x$. It also includes for example $(5, x+2), (5, x+3)$ and we can think of this closure as a branched curve going through these points.
We can think of this as the fibration having both horizontal and vertical directions. In the vertical one, we have the affine schemes Spec$\mathbb{F}_p[x]$, Spec $\mathbb{Q}[x]$, whereas in the horizontal, we have the closures given by the closed affine subschemes which look for example like Spec,$\mathbb{Z}[x]/(x^2 + 1) \simeq$ Spec $\mathbb{Z}[i]$. In fact, this affine scheme has three different points and is also fibered over Spec $\mathbb{Z}$ - these are the ramifications points $2 = (1+i)^2(-i)$, the primes which stay irreducible (the 3 mod 4 primes) and the primes which split (1 mod 4). Note also that Pic Spec $\mathbb{Z}[i]$ is the ideal class group!

Example (Segre embedding): The set of 2×2 matrices over a field k with rank less than 2 are given by the vanishing of the determinant, i.e. the affine scheme Spec$\, k[x, y, z, w]/(xy - zw)$. When we projectivize, we get $\mathbb{P}^1 \times \mathbb{P}^1$ using the Segre embedding!

3.2.5 Schemes and their basic properties

Definition 3.7 (Schemes): *Schemes are defined as locally ringed spaces* $(X, \mathcal{O}_X)$ *such that each point has a neighbourhood* U*and an isomorphism* $(U, \mathcal{O}_U) \simeq (\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}(A)})$ *as locally ringed spaces.*

Note that for a scheme $(X, \mathcal{O}_X)$and a point $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring, since p has an affine neighbourhood Spec$\, A$ where we are identifying it with a prime $\mathfrak{p}$ and the stalk is $A_{\mathfrak{p}}$.

This motivates the definition of a morphism of schemes as morphisms of locally ringed spaces:

Definition 3.8 (Morphisms in Sch):

A morphism of schemes consists of the data of a continous map $f : X \to Y$ and a morphism of sheaves $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ (equivalently, a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ by adjointness), which induces maps on stalks: $f_{\mathfrak{p}} : \mathcal{O}_{Y,f(\mathfrak{p})} \to \mathcal{O}_{X,\mathfrak{p}}$

In this category, the spectrum of the zero ring is initial, and $\operatorname{Spec}\mathbb{Z}$ *is terminal.*

Remark: note that the induced map on stalks follows immediately from the definition using the inverse image sheaf, as $(f^{-1}\mathcal{O}_Y)_{\mathfrak{p}} = \mathcal{O}_{Y,f\mathfrak{p}}$. However, if we use the direct image, we first get an induced map on direct limits: $\varinjlim \mathcal{O}_Y(V) \to \varinjlim \mathcal{O}_X(f^{-1}V)$, where the limit ranges over all open subsets V of $f\mathfrak{p}$. This means that $f^{-1}V$ ranges over only a subset of the opens around $\mathfrak{p}$ and we must further use a map $\varinjlim \mathcal{O}_X(f^{-1}V) \to \varinjlim \mathcal{O}_X(U) = \mathcal{O}_{X,\mathfrak{p}}$.

Example (Projective line): We are gluing the two affine rings along their affine open subsets $U = \operatorname{Spec} k[t] - \mathbf{V}(t) = \mathbf{D}_t \simeq \operatorname{Spec} k[t, t^{-1}]$, $V = \operatorname{Spec} k[u] - \mathbf{V}(u) = \mathbf{D}_u \simeq \operatorname{Spec} k[u, u^{-1}]$ along $t \mapsto u^{-1}$ (think of plane minus origin). In other words, we are forming a pushout, obtained from the maps $u \mapsto v^{-1}, t \mapsto v$ of rings.

$$\begin{tikzcd}
\operatorname{Spec} k[v, v^{-1}] \arrow[r] \arrow[d] & \operatorname{Spec} k[t] \arrow[d] \\
\operatorname{Spec} k[u] \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

Passing to global sections, we obtain a pullback of rings:

$$\begin{tikzcd}
k[v, v^{-1}] & k[t] \arrow[l] \\
k[u] \arrow[u] & \Gamma(\mathbb{P}^1) \arrow[l] \arrow[u]
\end{tikzcd}$$

Hence, the global sections of $\mathbb{P}^1$ must be the constant functions, being the intersection of these two rings inside $k[v, v^{-1}]$.

Now, let's consider the same thing, but with $\mathbb{Z}$ and think about the $\mathbb{Z}$-valued points $\mathbb{P}^1_{\mathbb{Z}}(\mathbb{Z})$ given by morphisms $\operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1_{\mathbb{Z}}$. The problem is that the image of such a morphism might not land in any open affine, so we can consider the preimages of the open cover of projective space. This will consists of two opens in $\operatorname{Spec} \mathbb{Z}$, which generally look like $\operatorname{Spec} \mathbb{Z}[1/n]$. To create a morphism, we glue two morphism as in the diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{Z}[1/n, 1/m] \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[1/m] \arrow[d] \arrow[rrd, bend left] & & \\
\operatorname{Spec} \mathbb{Z}[1/n] \arrow[r] \arrow[rrd, bend right] & \operatorname{Spec} \mathbb{Z} & \operatorname{Spec} \mathbb{Z}[x, x^{-1}] \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[x^{-1}] \arrow[d] \\
& & \operatorname{Spec} \mathbb{Z}[x] \arrow[r] & \mathbb{P}^1_{\mathbb{Z}}
\end{tikzcd}$$

Supposing that $nm \neq 0$, since the two opens cover $\operatorname{Spec} \mathbb{Z}$, we must have that $(n, m) = 1$. We have induced morphisms $\mathbb{Z}[x] \to \mathbb{Z}[1/n], \mathbb{Z}[x^{-1}] \to \mathbb{Z}[1/m]$ given by sending $x \mapsto a/n^l, x^{-1} \mapsto b/m^k$ and hence this means that $ab = n^l m^k$.

Example (Continued): From Patrick Da Silva's comment on a youtube video: "You can replace $D(m)$ by $D(m^k)$ and assume WLOG that $k = l = 1$ because it doesn't affect the map, just how you look at it. Using the fact that $\mathbb{Z}$ is a UFD, you can assume that $(a,m) = gcd(b,n) = 1$ (because again, this doesn't affect your map. So the equality $ab/mn = 1$ means $ab = mn$ and since $(a,m) = 1$, a divides n, and since $(b,n) = 1$, b divides m. So in any case a/b and b/a are units. Since $ab/mn = 1$, we can write $m = bu$ and $n = av$ where u, v are units, and $ab/mn = 1$ gives $1/uv = 1$, i.e. $uv = 1$. Writing $a/m = a/bu$ and $b/n = b/av = bu/a$, we see that the map $\mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$ corresponds to the pair $(a/bu, bu/a)$, i.e. an arbitrary non-zero rational number. It is a crucial theorem for morphisms to projective space that for a Y-scheme X, a Y-morphism $X -> \mathbb{P}^1_Y$ is determined by an invertible sheaf L on X and an invertible global section of L. Invertible sheaves over UFD correspond to rational numbers, and a global section of the invertible sheaf corresponding to a/b is just a/b times a unit of $\mathbb{Z}$! All in all, we such morphisms are in bijection with pairs $[i : j]$ of coprime integers up to a unit scaling. This same procedure works for all UFD's, where the Picard group (i.e. the ideal class group) is trivial. However, for non-UFD's it does not necessarily work.

Definition 3.9 (Reducibility and integrality): *X is reduced if all its open affines correspond to reduced rings, and similarly X is integral if the rings are integral domains. It is a fact that X is integral if it is reduced and irreducible.*

Definition 3.10 (Function field): *For an integral scheme, the local ring at the generic point is called the function field and is denoted $K(X)$. For every affine open* Spec *A, the fraction field of A is the function field.*

Proposition 3.11 (Reduced morphisms factor through closed image): *A morphism $f : X \to Y$ with X reduced factors through a closed $Z \subseteq Y$ if and only if $f(X) \subset Z$ set theoretically.*

Proof. See `https://stacks.math.columbia.edu/tag/0356`. ☐

Definition 3.12 (Noetherian schemes): *A scheme is locally Noetherian if there is a cover by open affines with global sections noetherian rings. A scheme is Noetherian if it is quasicompact and locally Noetherian. As is usual, this is a local property and we have that X is Noetherian if every open affine is Noetherian.*

Definition 3.13 (Finite type and finite morphisms): *A morphism of schemes $f : X \to Y$ is of locally finite type if there is an open affine cover $\{\mathrm{Spec}\, B_i\}$ of Y such that the preimage is covered by open affines $f^{-1} Spec B_i = \cup \mathrm{Spec}\, A_{ij}$ with each ring A_{ij} being finitely generated as an algebra over B_i. It is of finite type if this cover $\mathrm{Spec}\, A_{ij}$ can be taken to be finite.*
f is finite if there exists a cover such that $f^{-1} Spec B_i = Spec A_i$ and A_i is finitely generated as a B_i-module. This means that the fibers form finite discrete sets.

Note that Noetherianness and locally finite type, as well as quasicompactness, are local properties on the codomain. Hence, since finite type = quasicompact + locally finite type, that is also a local property (i.e. every open affine in Y can be covered by a finite number of open affines whose preimages have a cover by algebras which are f.g.).

Definition 3.14 (Immersions and dominant maps):
$f : X \to Y$ is called an open immersion if it induces an isomorphism onto an open subscheme $(U, \mathcal{O}_Y|_U)$ $g : X \to Y$ is a closed immersion if topologically it is a homeomorphism onto a closed subset of Y and $g^\#$ is a surjective map on sheaves. $f : X \to Y$ is dominant if $f(X)$ is dense in Y.

Remark: Surjective ring homomorphisms induce closed immersions on affine schemes, since they look like $R \to A = R/I$, and $\mathbf{V}(I)$ is closed. Note that surjectivity for sheaves is stalk local, so to check map on sheaves is surjective, we just need to show the maps $R_\mathfrak{p} \to A_\mathfrak{q}$ are surjective, where $\mathfrak{p} = f^{-1}\mathfrak{q}$. But this is true by surjectivity of f.

Proposition 3.15 (Closed immersons are stable under base change): *Closed immersions are stable under base change:*

$$\begin{tikzcd} X \times_S Y \arrow[r, tail] \arrow[d] & Y \arrow[d] \\ X \arrow[r, tail] & S \end{tikzcd}$$

In particular, they are universally closed.

Proof. We will have to use the fact that closed immersions are defined by the property that for any affine $\mathrm{Spec}\, A \subset S$, its preimage is an affine $\mathrm{Spec}\, A/I$ (since this is what all closed subschemes of affines look like. Need quasicoherent sheaves to do this, or Hartshorne's affineness criterion). In particular, closed immersions are affine! Then this reduces to a local computation, and the fact that $p_1^{-1}(U) = U \times_S Y$, as well as that $B \to B/IB$ is surjective. $\square$

Corollary 3.16 (Closed immersions are proper): *We have just seen that closed immersions are universally closed. Moreover, the diagonal map $X \to X \times_S X$ is just the identity, which is a closed immersion, which means they are separated as well. Finally, for any affine $\mathrm{Spec}\, A \subset S$, the preimage is the affine $\mathrm{Spec}\, A/I$ which is finitely generated, so it is also finite type. All in all, closed immersions are proper.*

We have the following properties of finite type morphisms:

- Closed immersions are finite type, and moreover finite
- Quasicompact open immersions are finite type
- Finite type compositions are finite type
- Finite type morphisms are stable under base change
- if $X \to Y$ is finite type with Y Noetherian, then X is Noetherian
- Finite type morphisms are proper (this will be proved later). So we could also have just said: closed immersions are finite, and hence proper.

Remark: Injective maps on rings induce dominant maps on the associated affine schemes, since the image will contain the generic point. The opposite is also true:

Proposition 3.17 (Dominant maps between affine schemes): Spec $A \to$ Spec B *is a dominant map of affine schemes if and only if the associated ring map is injective.*

Proof. One side was just mentioned. Conversely, assume the generic point of B is contained in the image, i.e. there is some prime ρ of A such that $f(\rho) = (0)$ or in other words $\varphi^{-1}(\rho) = (0)$ where φ is the associated ring map $B \to A$. Then we get maps of local rings $B_{(0)} \to A_\rho$, where the left is Frac(B). Assume $\varphi(b) = 0$. Then, either $b = 0$, which is okay, or $b \neq 0$, in which case $\varphi(b) \notin \rho$ and then the induced map on the local rings will be nonzero, as when we pass to fraction fields we get an injection (maps between fields are always injective). Hence, φ is injective. $\square$

A closed subscheme X of Y is an equivalence class of closed immersions $X \to Y$, where two closed immersions are equivalent if there is a commutative triangle:

$$\begin{tikzcd} X \arrow[rr, "\simeq"] \arrow[dr, tail] & & X' \arrow[dl, tail] \\ & Y & \end{tikzcd}$$

Note that closed immersions are monic (in fact, the proper monomorphisms are precisely the closed immersions), so this gives a notion of a subobject in the category of schemes.

In fact, using the criterion for affineness one can show that any closed subscheme of an affine scheme is affine (note that the same statement about open subschemes is false) (Hartshorne, 3.11)

3.2.6 The Proj construction

Given a graded ring $A = \bigoplus A_i$, we want to associate a schematic object to it. We are using the convention that the grading is in nonnegative integers and that $A_+ = \bigoplus_{i \geq 1} A_i$ is the irrelevant ideal.

Definition 3.18 (Proj):

$$\operatorname{Proj} A = \{\textit{homogenous prime ideals not containing } A_+\}$$

$$\mathbf{V}(I) = \{P \in \operatorname{Proj} A | I \subset P\}$$

Now, if $f \in A_1$ and A is generated by degree one elements, then we can define a basic open

$$\mathbf{D}_f \leftrightarrow \textit{Homogenous nonirrelevant primes of } A[1/f] \leftrightarrow \textit{All primes in } A[1/f]_0$$

(Refer to Vakil for some of this stuff.)

Basically, for a homogenous prime not containing f, associate its localization as usual, and then intersect with degree 0 part to get a prime in $A[1/f]_0$. Conversely, given a prime $\mathfrak{p}$ in $A[1/f]_0$, one can clear all the f denominators to get a prime in A which does not contain f, i.e. $\mathfrak{q} = \{a | a/f^n \in \mathfrak{p}\} \subset A$. This allows us to identify $\mathbf{D}_f \simeq \operatorname{Spec} A[1/f]_0$.

Now, note that in this case, we still have $\mathbf{D}_f \cap \mathbf{D}_g = \mathbf{D}_{fg}$. Under the identifications, we see firstly that $\operatorname{Spec} A[1/fg]_0 \simeq \mathbf{D}_{fg}$, which is an open subset of $\mathbf{D}_f \simeq \operatorname{Spec} A[1/f]_0$. In fact, this identifies $\operatorname{Spec} A[1/fg]_0$ as an open subset of $\operatorname{Spec} A[1/f]_0$, sending a prime in the first to a prime in the second by clearing out all the g's in the denominators.

$$\begin{tikzcd}
\operatorname{Spec} A[1/fg]_0 \arrow[d] & \mathbf{D}_{fg} = \mathbf{D}_f \cap \mathbf{D}_g \arrow[l, "\sim"'] \arrow[d] \\
\operatorname{Spec} A[1/f]_0 & \mathbf{D}_f \arrow[l, "\sim"']
\end{tikzcd}$$

This also realizes $\operatorname{Spec} A[1/fg]_0$ as $\mathbf{D}(g/f) \subset \operatorname{Spec} A[1/f]_0$, since $A[1/fg]_0 \simeq A[1/f]_0[(g/f)^{-1}]$, so primes in the first correspond to primes in the second not containing g/f, and this allows us to define a scheme structure on the basis of distinguished opens, i.e. $\operatorname{Proj} A$ is affine in the distinguished open neighbourhoods, and the isomorphism are coherent.

Remark: To get coordinates on $\mathbb{P}^n_k$, consider a point in usual projective space with homogenous coords $\alpha = [\alpha_0 : \ldots : \alpha_n]$. We get a homogenous ideal $\rho(\alpha)$ generated by the elements $(\alpha_j x_i - \alpha_i x_j)$, and it is prime because the quotient is $k[x_0, \ldots, x_n]/\rho(\alpha) = k[x_0]$. In fact, this gives us precisely the rational points of $\mathbb{P}^n_k$, i.e. ones with residue field k. To see this, note that since $\rho(\alpha)$ is not the irrelevant ideal and contains each $x_i - x_0\alpha_0\alpha_i^{-1}$, then $\rho(\alpha) \in \mathbf{D}_{x_0}$. This corresponds to an ideal in $k[x_1/x_0, .., x_n/x_0]$ generated by $x_i/x_0 - \alpha_0\alpha_i^{-1}$ and hence has $\kappa(\rho(\alpha)) = k$. To show surjectivity, if we have some rational point x, it is in some distinguished open, WLOG $\mathbf{D}_{x_0}$. Hence, we can take $x_i/x_0 \in \mathcal{O}(\mathbf{D}_{x_0})$ and look at its image in the stalk at x, which is an element $\alpha_i \in \kappa(x) = k$. Then $[\alpha_0 : \ldots : \alpha_n]$ is an element with $\rho(\alpha) = x$. (see Liu for more on this)

3.2.7 A few words on irreducible components and integral schemes

Any Noetherian scheme X decomposes into a union of finitely many irreducible components, each of which corresponds to a generic point. On affine schemes, generics correspond to minimal prime ideals.

Definition 3.19 (Specialization): *A point x specializes to y if $y \in \overline{\{x\}}$*

We have a bijection between irreducible components of X and generic points, by taking a generic point and looking at its closure. Moreover, this induces a bijection between irreducibles passing through a point x and the irreducible components of $\operatorname{Spec} \mathcal{O}_{X,x}$. This is because these correspond to minimal primes of the local ring, which affine locally looks like A_p i.e. correspond to minimal primes in A containing a prime p, which in turn corresponds to generics specializing to p, since $y \leftrightarrow \mathfrak{q}$ specializes to $x \leftrightarrow \mathfrak{p}$ if and only if $\mathfrak{q} \subseteq \mathfrak{p}$.

A scheme is integral if every ring of sections is an integral domain. One can show that a scheme is integral iff it is reduced and irreducible, and hence integral schemes have a unique generic point. Moreover, the field of fractions of any ring of sections is equal to the residue field of the generic point, which is called the field of rational functions on X and denoted $K(X)$. We can think of any ring of sections and stalk as sitting inside of this field of fractions and then we have the identification $\mathcal{O}_X(U) = \cap_{x \in U} \mathcal{O}_{X,x}$.

Remark: the generic point of a closed subscheme $Y \subset X$ is not the same as the generic point of X! Example: $X = \mathbb{A}^n = \operatorname{Spec} k[x_1, ..., x_n]$, $Y = \mathbb{V}(x_1) = \operatorname{Spec} k[x_2, ..., x_n]$, then the generic point in Y will be the ideal (x_1) in X. However, it is true for open subschemes. More generally, a closed subscheme of an affine scheme $\operatorname{Spec} A$ is given by $\operatorname{Spec} A/I$ and its generic point will correspond to the ideal I.

3.2.8 Adjointness of global sections and Spec

We have two contravariant functors going in two directions: $\operatorname{Spec} : \text{Rings} \to \text{Locally Ringed Spaces}$ and $\Gamma : \text{Locally Ringed Spaces} \to \text{Rings}$. In fact, they are adjoint!

$$\operatorname{Hom}_{Rings^{op}}(\Gamma S, R) \simeq \operatorname{Hom}_{LRS}(S, \operatorname{Spec} R)$$

To show this, given a ring homomorphism $f : R \to \Gamma S$, we construct $\hat{f} : S \to \operatorname{Spec} R$ as follows:

$$\hat{f}(s) = \{r \in R | \overline{f(r)} \in \mathfrak{m}_s \subset \mathcal{O}_{S,s}\}$$

In other words, we collect all elements of R, the germs of whose images lie in the unique maximal hence prime ideal of the local ring $\mathcal{O}_{S,s}$. This is easily seen to be a prime ideal. Moreover, this is a continous map: if $g \in R$, then $\hat{f}^{-1}\mathbf{D}_g = \{s \in S | \overline{f(g)} \notin \mathfrak{m}_s\}$. This means that $f(g)$ is invertible in the local ring of s, i.e. there is a t such that $\overline{f(g)t} = 1$. But this holds in the stalk, hence holds in a small neighbourhood of s by definition, hence $f(g)$ is invertible around s, showing continuity.

On stalks, the map $f : R \to \Gamma S$ induces a map $R_{\hat{f}(s)} \to \mathcal{O}_{S,s}$, since anything whose image does not land in $\mathfrak{m}_s$ can be inverted and the image would still be defined, as everything not in $\mathfrak{m}_s$ is invertible. It is also a local ring map: it sends the maximal ideal $\hat{f}(s)R_{\hat{f}(s)}$ into $\mathfrak{m}_s$ by definition! To have a map of sheaves, we want a map:

$$\mathcal{O}_{\operatorname{Spec} R}(\mathbf{D}_f) \to \mathcal{O}_S(\hat{f}^{-1}\mathbf{D}_g)$$

The open set $\hat{f}^{-1}\mathbf{D}_g = V$ consists of the s where $f(g)$ becomes a unit in the stalks. But we have ring homomorphisms $R \to \mathcal{O}_S(S) \xrightarrow{res} \mathcal{O}_S(V)$ and this basically lifts to a map $R[1/g] \to \mathcal{O}_S(V)$ by the property of V (g becomes a unit in the sections over V). But $R[1/g]$ is precisely $\mathcal{O}_{\operatorname{Spec} R}(\mathbf{D}_f)$, so it works out.

To check that the compositions are natural isomorphisms, one side is easy, but the other uses in an essential way that the maps are maps of locally ringed spaces, i.e. if $\varphi : S \to \operatorname{Spec} R$, then we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{\text{Spec } R}(\text{Spec } R) \arrow[r, "\varphi^\#"] \arrow[d] & \mathcal{O}_S(S) \arrow[d] \\
\mathcal{O}_{\text{Spec } R, \varphi(s)} \arrow[r, "\varphi^\#"] & \mathcal{O}_{S,s}
\end{tikzcd}$$

In other words, we get a map of rings $\varphi^\# : R \to \Gamma S$ and we want to check it induces φ.

But $\hat{\varphi^\#}(s) = \{r \in R | \overline{\varphi^\#(r)} \in \mathfrak{m}_S \subset \mathcal{O}_{S,s}\}$, so if we denote $\varphi(s) = \mathfrak{p}$, we see that $\mathfrak{p} \subset \hat{\varphi^\#}(s)$. Conversely, if $r \in R$ is in $\hat{\varphi^\#}(s)$, then $\overline{r} \in {\varphi^\#}^{-1}\mathfrak{m}_s = \mathfrak{p}R_{\mathfrak{p}}$ and hence $r \in \mathfrak{p}$. So $\varphi(s) = \mathfrak{p} = \hat{\varphi^\#}(s)$.

Now note the following category theoretic fact: any adjunction induces an equivalence on a certain subcategory:

[Exercise 3.14, 3.16](https://www.notion.so/Exercise-3-14-3-16-4a8910d7cb3e4f4c8271f5e41c2d3f29)

Hence, we get an equivalence between the opposite of the category of rings and the category of affine schemes!

Proposition 3.20 (Corollary): $\operatorname{Hom}(X, \mathbb{A}^1) \simeq \operatorname{Hom}(\mathbb{Z}[t], \mathcal{O}_X(X)) \simeq \mathcal{O}_X(X)$, *i.e. affine space represents the global sections functor! Similarly, affine n-space represents the functor of n functions, and invertible functions are represented by* $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$, *i.e. the global sections of* $\mathcal{O}_X^\times$.

3.2.9 A criterion for affineness

Similar to how we defined a map of locally ringed spaces from maps of rings, we can define, for any scheme, a generalized open subset. This generalizes the distinguished opens in affine schemes. Take $f \in \mathcal{O}_X(X)$ a global section and define:

$$X_f = \{x \in X | f_x \notin \mathfrak{m}_x\}$$

Then this is an open subset of X and in fact its intersection with any affine open $U \subseteq X$ is precisely a distinguished open in the classical sense, i.e. $X_f \cap U = \mathbf{D}(\overline{f})$, where $\overline{f}$ is the image of f under the restriction map. One can in fact show more: $\mathcal{O}_X(X_f) = [\mathcal{O}_X(X)]_f$, just like in the usual case when $X = \operatorname{Spec} A$ we have $\mathcal{O}_X(X_f) = A_f$. This also implies a criterion of affineness:

Proposition 3.21 (Affineness criterion): *A morphism between schemes is an isomorphism if and only if it restricts to an isomorphisms on all opens. X is affine iff there are finitely many global sections $f_1, .., f_r$ which generate the whole ring of global sections, and furthermore each X_{f_i} is affine. (This is all in Hartshorne, 2.16, 2.17).*

We define a morphism of schemes to be affine if there is an open affine cover of the target such that the preimages are affine. This property is local.

Corollary 3.22 (Affine morphisms are local): *A morphism between schemes is affine if, for all open affine $V \subset Y$, the preimage $f^{-1}(V)$ is affine.*

Proof. We can reduce to the case $Y = \operatorname{Spec} A$ in which case we want to show X is also affine. Affine schemes are quasicompact, so there is a cover $\operatorname{Spec} A = \bigcup \mathbf{D}(a_i)$ for a finite set of a_i which generate the unit ideal such that the preimages are affine in X. But then we can cover X by these affines

$$X_i := \{x \in X | f(x) \in \mathbf{D}(a_i)\} = f^{-1}(\mathbf{D}(a_i)) = \{x \in X | (f^\#(a_i))_x \notin \mathfrak{m}_x\}$$

which impies that X is affine, by the affineness criterion. □

Proposition 3.23 (Affine maps preserve cohomology): *Affine maps between separated schemes have the property that $H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F})$. This can be seen by using Cech cohomology, since the map will preserve Cech covers. An important example is the inclusion map, which can be applied for example to subvarieties in $\mathbb{P}^n$.*

3.2.10 The functor of points

Definition 3.24 (Functor of points): *A k-valued point in a scheme X is a map* $\operatorname{Spec} k \to X$. *We denote the set of these points as* $X(k) := \operatorname{Hom}(\operatorname{Spec} k, X)$. *More generally, this defines a functor of points:*

$$\text{Rings} \to \text{Set}$$

$$R \to X(R)$$

This is the composition of Spec *with* $\operatorname{Hom}(-, X)$.

This idea of thinking of a scheme as the functor it represents is the core of the Yoneda philosophy. We will see that projective space represents the functor which to a scheme X associates the set of line bundles with $n+1$ sections with empty common zero locus. More generally, one can also define the Grassmanian as the functor which sends X to the set of equivalence class of surjections $\mathcal{O}_X^n \to \mathcal{D}$. Refer to Vakil 16.7.

3.2.11 Fibre products

Recall that Spec is right adjoint to global sections, so it preserves limits. This means that $\operatorname{Spec}(A \otimes_C B) = \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$, since tensor products are pushouts in the category of rings, but pullbacks in the opposite of the category of rings, which is the domain of Spec. This shows that fibre products of affine schemes exist and are also affine schemes. This will be enough, using some combinatorial reasoning, that fibre products exist for all schemes, and this is the "correct" notion of product (It is the product in the category Schemes $\downarrow S$). For the full argument, consult Hartshorne. The idea is the following: for open subschemes U of affine schemes X, we can take $p_X^{-1}(U) = U \times_S Y$, since it satisfies the universal mapping property, where $p_X : X \times_S Y \to X$. For the general case, we have to basically glue schemes together. Note that is S is covered by affines S_i and if their preimages are opens X_i, Y_i, then $X_i \times_{S_i} Y_i$ exists and is equal to $X_i \times_S Y_i$.

Example (Examples): if X_1, X_2 are closed subschemes of Y, then their fibre product is their "intersection". The fibre of $y \in Y$ is a subscheme of X.

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, tail] \arrow[d, tail] & X_1 \arrow[d, tail] \\ X_2 \arrow[r, tail] & Y \end{tikzcd} \qquad \begin{tikzcd} \pi^{-1}(y) \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \{y\} \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}(\kappa(B \otimes \mathfrak{p})) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec}(\kappa(\mathfrak{p})) \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

To see why $\{y\}$ is a closed subscheme of Y, note that we are identifying it with the affine scheme $\operatorname{Spec} \kappa(y)$ (recall that a map from the spectrum of a field K to a scheme Y is given precisely by an element y whose residue field is included in K and apply this to $K = \kappa(y)$) In the affine case, this corresponds to having $Y = \operatorname{Spec} A, X = \operatorname{Spec} B, y = \mathfrak{p}$ and then we have the pushout tensor product of rings $B \otimes_A \kappa(\mathfrak{p})$, which on the ring level corresponds to the fiber over $\mathfrak{p}$ which is exactly $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$.

Note the important magic square (Vakil's terminology), which realizes the fiber product as a pullback along the diagonal, i.e. we're intersecting with the diagonal.

$$\begin{tikzcd} (X \times Y) \times_{S \times S} S \arrow[r] \arrow[d] & X \times Y \arrow[d, "f \times g"] \\ S \arrow[r, "\Delta"] & S \times S \end{tikzcd}$$

Remark: similarly to the case of affines, the fibred product of two projective schemes defined using Proj over A algebras is as follows:

$$\operatorname{Proj}(B \otimes_A C) \simeq \operatorname{Proj} B \times_{\operatorname{Spec} A} \operatorname{Proj} C$$

Example (Intersection of two affines): Take the parabola $\operatorname{Spec}\mathbb{C}[x,y]/(y-x^2)$ and the x-axis $\operatorname{Spec}\mathbb{C}[x,y]/(y)$. Their fibre product is given by the Spec of the tensor product $\operatorname{Spec}\mathbb{C}[x]/(x^2)$ and is given by the double point where they intersect. If we perurb the x-axis a bit to $\operatorname{Spec}\mathbb{C}[x,y]/(y-\epsilon)$, we instead get $\operatorname{Spec}\mathbb{C}[x]/(x^2-\epsilon) = \operatorname{Spec}\mathbb{C}[\sqrt{\epsilon}]\coprod\operatorname{Spec}\mathbb{C}[-\sqrt{\epsilon}]$ which is two single points.

3.3 Morphisms between schemes

3.3.1 Separated morphisms

Since schemes are never Hausdorff, we need a replacement for that notion, which is separatedness.

Definition 3.25 (Separated morphisms): *A map $f : X \to S$ is separated if the diagonal map is a closed immersion. This is the map induced from $(X, 1_X, 1_X)$ to $(X \times_S X, f, f)$:*

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/S}"] & X \times_S X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & S \end{tikzcd}$$

Example (Example): If X, S are affine, then $X = \operatorname{Spec} A, S = \operatorname{Spec} R$ and the map is induced by a map of rings $R \to A$. Hence, the fiber product is just $\operatorname{Spec}(A \otimes_R A)$ and the diagonal map is the one induced by multiplication. But $A \otimes_R A \xrightarrow{mult} A$ is surjective, so gives a closed immersion when we take Spec.

Remark: a map is separated if and only if the weaker condition that the image of the diagonal map is closed is satisfied. Moreover, an image of a quasicompact map is closed iff it is stable under specialization. Furthermore, separated morphisms are stable under base change.

Proposition 3.26 (Epi-monic factorization of diagonal map): *The map $\Delta_{X/S}$ can be factored through a closed immersion μ and open immersion ν:*

$$X \xrightarrow{\mu} U \xrightarrow{\nu} X \times_S X$$

Note that open and closed immersions are separated, since then the fibre product is just X itself, and so is the image of the diagonal map. Furthermore, separated morphisms are stable under pullback (base change).

Proposition 3.27 (Intersections of affines in separated schemes): *In a separated scheme, intersections of affine subschemes are affine.*

Proof. Consider the diagram

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

By assumption, Δ is a closed immersion and hence the top arrow is a closed immersion, since they are stable under base change. But $U \times_S V$ is affine and closed subschemes of affines are also affine.

(I don't remember where we proved this - it is in the exercises in Hartshorne, or can be proved using ideal sheaves). $\square$

Proposition 3.28 (Morphisms agreeing on dense opens): *Let $X \to S$ be reduced, $Y \to S$ be separated, $f, g : X \to Y$ two S-morphisms agreeing on a dense open subset. Then $f = g$.*

Proof. Consider the diagram:

$$\begin{tikzcd}
X \arrow[dr, "{(f,g)}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& Y \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

We see that $(f, g)|U$ lands inside of $\Delta(Y)$. By 3.11, we only need to show that (f, g) lands inside $\Delta(Y)$ set theoretically, which is true, since U is dense and $\Delta(Y)$ is a closed subset, since Y is separated. $\square$

3.3.2 Separatedness of projective space

We want to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated, i.e. that the diagonal map induced by the identity maps is a closed immersion.

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\Delta"] & \mathbb{P}^n \times_{\operatorname{Spec} A} \mathbb{P}^n \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\
& \mathbb{P}^n \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

Let's use the affine open cover $U_i = \mathbf{D}(x_i) = \operatorname{Spec} A[x_0, \dots, x_n][1/x_i]_0$ and see what happens on $U_i \times U_j$. We claim that $\Delta^{-1}(U_i \times_{\operatorname{Spec} A} U_j) = U_i \cap U_j$. But this is clear, as if something in $\mathbb{P}^n$ were to map via the identity to both U_i and U_j then it must be in $U_i \cap U_j$. But now we are dealing only with affine things, and to show something is a closed immersion one only needs to check a ring homomorphism is surjective. In our case, $U_i \cap U_j \simeq \operatorname{Spec} A[x_0/x_i, \dots, x_n/x_i][x_i/x_j]_0$, whereas $U_i \times_{SpecA} U_j \simeq \operatorname{Spec}(A[x_0/x_i, \dots, x_n/x_i]_0 \otimes_A A[x_0/x_j, \dots, x_n/x_j]_0)$ and the associated map on rings $A[x_0/x_i, \dots, x_n/x_i]_0 \otimes_A A[x_0/x_j, \dots, x_n/x_j]_0 \to A[x_0/x_i, \dots, x_n/x_i][x_i/x_j]_0$ is obviously surjective, and we are done.

3.3.3 Proper maps

This notion is an analogue of compactness.

Definition 3.29 (Proper maps): *$f : X \to Y$ is of finite type if for any open affine* $\operatorname{Spec} A$ *in Y, the pre-image under f can be covered by finitely many affine opens* $\operatorname{Spec} B$ *with B a finitely generated A-algebra. A morphism is universally closed if any pullback of f is closed (as a topological map) - note that this includes f itself. f is proper if it is separated + universally closed + finite type*

Example (Example): $\mathbb{A}^1$ is closed but not universally closed! The base change with itself gives the affine plane, and the projection of the hyperbola $xy = 1$ gives the affine line minus the origin, which is not closed. Projective space is proper, as are all its closed subschemes.

Here is the analogy with compactness and sequences converging: we would like, given a curve C and point P and a morphism $C - P \to X$, to be able to extend this in at most one way if the scheme is separated. Locally, we can replace the curve by its local ring at P, a DVR (which is just a local PID). We can think of this as a thickening of Spec $K$and this scenario is represented in the valuative criterion:

Proposition 3.30 (Valuative criteria): *Assume X is Noetherian.*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

f is separated if and only if, for any DVR R and $K = FF(R)$ and a commuting diagram of the type above, there is at most one way of lifting the map $\operatorname{Spec} R \to Y$ *to X. f is universaly closed if and only if there is at least one way of lifting the map* $\operatorname{Spec} R \to Y$ *to X. f is proper if there is exactly one lift and it is of finite type.*

We need to think of what $\operatorname{Spec} K$ and $\operatorname{Spec} R$ represent, which comes in the following lemma:

Lemma 3.31 (Field and DVR valued points):

$$X(K) = \operatorname{Hom}(\operatorname{Spec} K, X) = \{\textit{points } x \in X, \kappa(x) \subseteq K\}$$

$$X(R) = \operatorname{Hom}(\operatorname{Spec} R, X) = \{\textit{points } x, y \in X, x \textit{ specializes to } y, \kappa(x) \subseteq K,$$
$$R \textit{ dominates the local ring at } y\}$$

Proof. $\operatorname{Spec} K$ is a one-point scheme, whose image is a point x. Moreover, we have a map on sheaves $\mathcal{O}_X \to f_* \mathcal{O}_{\operatorname{Spec} K}$ and on stalks $\mathcal{O}_{X,x} \to K$, which is the same as an inclusion $\kappa(x) \subseteq K$, proving the first bit.

For the second, $\operatorname{Spec} R$ has two points, the closed maximal ideal $\mathfrak{m}$ and the generic point. These have images $y, x \in X$, and in fact the morphism lands in the closed subscheme $\overline{\{x\}}$ (by Lemma 3.11 since R is reduced), which y is a member of, i.e y is a specialization of x . We have a local map from $\mathcal{O}_y$ to R compatible with $\kappa(x) \subseteq K$, which means that R dominates the local ring. □

We also need another lemma:

Lemma 3.32 (Closed images of quasicompact maps): *$f : X \to Y$ quasicompact has $f(X)$ closed in Y if and only if it is closed under specialization.*

Proof. One side is obvious, so need to show that if it is closed under specialization, then it is closed. Reduce to the case that $Y = \overline{f(X)}$ and Y affine. Given $y \in Y$, it must be in some $Y_i = \overline{f(X_i)}$, where $\cup X_i = X$ is a finite affine open cover of X, by quasicompactness of f. Put $X_i = \operatorname{Spec} A, Y_i = \operatorname{Spec} B$ (it is affine as it is a closed subset of an affine). The map f restricted to X_i and Y_i is dominant (denseness of image), hence it is injective on sheaves, i.e. $B \to A$ is injective, by 3.17. Now, y is given by some prime $\mathfrak{p} \subset B$, which moreover contains a minimal prime $\mathfrak{p}'$, given by a point y' which specializes to y, i.e. $\mathfrak{p} \in \mathbf{V}(\mathfrak{p}') = \overline{\{\mathfrak{p}'\}}$. We will show y' is in the image of f and conclude that y is also in the image of f, as by assumption it is closed under specialization.

But the fibre of y' is given by the affine scheme $\operatorname{Spec}(A \otimes \kappa(\mathfrak{p}')) = \operatorname{Spec}(A \otimes B_{\mathfrak{p}'})$ since $B_{\mathfrak{p}'}$ is a field, and any prime ideal in the inverse image of the localization map $A \to A \otimes B_{\mathfrak{p}'}$ will give us an element x' with $f(x') = y'$. This completes the proof. □

We now prove the valuative criterion of separatedness.

Proof. Valuative criterion of separatedness Assume that f, hence that $\Delta(X)$ is closed inside $X \times_Y X$. Consider two $h, h' : \operatorname{Spec} R \to X$ that make the diagram commute. This induces a map $h'' : \operatorname{Spec} R \to X \times_Y X$ and moreover, since h and h' are the same on $\operatorname{Spec} K$, they send the generic point to the same thing, hence its image lies in $\Delta(X)$. By assumption, this is closed, hence contains its closure, i.e. the other point of $\operatorname{Spec} R$, implying that $h = h'$.

Conversely, we want to show that $\Delta(X)$ is closed. By the fact that X is Noetherian and the diagonal map is quasicompact, we reduce to showing that it is closed under specialization by using the previouse lemma. Let $\zeta \rightsquigarrow \zeta'$ be a specialization with $\zeta \in \Delta(X)$. Hence, $\zeta' \in \overline{\{\zeta\}}$. Then the local ring at ζ' is contained in $k(\zeta)$, thinking inside the subscheme $\overline{\{\zeta\}}$. We can replace this local ring by some local ring R which dominates it and then by Lemma 3.31, we get a morphism $\operatorname{Spec} R \to X \times_Y X$ sending the generic point to ζ and the closed point to ζ'. Since $\zeta \in \Delta(X)$, composing with the projections gives two morphisms $\operatorname{Spec} R \to X$ giving the same morphism to Y and agreeing on $\operatorname{Spec} K$, since $\zeta \in \Delta(X)$. By the assumed condition of at most one lift, these two morphisms are the same and hence $\operatorname{Spec} R \to X \times_Y X$ factors through the diagonal $\Delta(X)$, implying that $\zeta' \in \Delta(X)$. □

Proof. Valuative criterion of properness The proof for properness follows along similar lines. The idea is as follows: if f is proper, we have at most one lift, so we need to show existence, which is done by doing base change of the morphism $f : X \to Y$ along $\operatorname{Spec} R \to Y$, resulting in an induced map $\operatorname{Spec} K \to X \times_Y \operatorname{Spec} R = X_R$. The base change projection f' is closed by universal closedness and hence one can show that f' sends the closed subscheme given by the closure $Z = \overline{\{\zeta\}}$, where ζ is the image of the unique point of $\operatorname{Spec} K$ in X_R, to the whole of $\operatorname{Spec} R$. This gives a local homomorphism from R to the local ring at another point of Z, which in fact must be an isomorphism and then we conclude again by invoking 3.31.

Conversely, if there exists a unique lift, we know it is separated, and we only need to show it is universally closed. Given $Y' \to Y$ with base change X', we need to show that $f' : X' \to Y'$ is closed and hence that it sends closeds to closeds. The Noetherian hypothesis implies f' is quasicompact when restricted to Z, so by 3.32, we reduce to showing that $f'(Z)$ is stable under specialization. Take $z_1 \in Z'$ with image y_1. Consider a specialization $y_0 \in \overline{\{y_1\}}$. By 3.31, we have that $\kappa(y_1) \subset \kappa(z_1)$ and moreover $FF(\mathcal{O}_{y_0}) = \kappa(y_1)$. Take $K = \kappa(z_1)$ and R a DVR replacement of $\mathcal{O}_{y_0}$ which dominates it. Again, by 3.31, we get morphisms $\operatorname{Spec} K \to Z, \operatorname{Spec} R \to Y'$ which make the diagram commute:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[urr, dashed] & Y' \arrow[r] & Y \end{tikzcd}$$

By the assumption of unique lifting, we get a map $\operatorname{Spec} R \to X$ and hence this factors through the fiber product X'. The generic point of $\operatorname{Spec} R$ goes to z_1 and Z is closed, so it in fact factors through $Z \subset X'$. Then the image of the closed point in $\operatorname{Spec} R$ will produce a point $z_0 \in Z$ with image y_0 in Y', and hence we are done. $\square$

Proposition 3.33 (Corollary):

- *Open and closed immersions are separated, and closed immersions are proper.*
- *Compositions of separated/proper morphisms are separated/proper*
- *Separated/proper morphisms are stable under base change. Moreover, products of separated/proper morphisms are separated/proper*
- *A morphism is separated iff the target can be covered by opens such that f restricts to a separated morphism on the preimages of these.*
- *Properness is local on the codomain.*

Proposition 3.34 (Finite morphisms are proper): *Finite morphisms are proper. (assuming X is Noetherian)*

Proof. Properness is local in the codomain, so we can reduce to the case $Y = \operatorname{Spec} A$. But then, we can use the valuative criteria:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dotted, "?"] & \operatorname{Spec} A
\end{tikzcd}$$

Recall 3.13 that a finite morphism is affine, in the sense that $f^{-1} \operatorname{Spec} A$ is an affine $\operatorname{Spec} B$ such that B if finitely generated as an A-module. So we are reduced to showing that there is a lift in the dual diagram:

$$\begin{tikzcd}
K & B \arrow[l] \arrow[dl, dotted, "?"] \\
R \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

But the generators of B over A will map to elements of K integral over R, but DVR's are integrally closed, so in fact those generators must land in R and hence a lift exists and is unique by construction.

□

> **Proposition 3.35 (Images of proper schemes are proper):** *Suppose $f : X \to Y$ is a morphisms of separated S-schemes of finite type, with S Noetherian. Let Z be a closed subscheme of X which is proper over S. Then $f(Z)$ is closed in Y and is proper over S*

Proof. The following is a Cartesian square:

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "{(f,1)}"] \\
Y \arrow[r, "\Delta"] & Y \times_S Y
\end{tikzcd}$$

I guess formally, one has to chase a lot of diagrams like

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] \arrow[rr, bend left, "{(f,1)}"] & Y \arrow[d] & Y \times_S Y \arrow[l] \arrow[d] \\
X \arrow[r] \arrow[u, bend left, "\Gamma_f"] \arrow[rr, bend right, "f"'] & S & Y \arrow[l] \arrow[u, bend right, "\Delta"']
\end{tikzcd}$$

to verify that it is indeed a pullback square (given $W \to Y, W \to X \times_S Y$, define $W \to X$ by composing with the projection and it all works out).

In any case, since Y is separated, Δ is a closed immersion and hence so is Γ_f. This shows the first part, since the projection map p_2 preserves closed sets. Now, $Z \to f(Z)$ is a surjective morphism with Z proper over S, hence universally closed. This implies that $f(Z)$ is universally closed over S by just considering any base change $S' \to S$. But in addition Y is separated and locally finite type, so so is $f(Z)$ and hence combining all the definitiones we see that $f(Z)$ is proper over S. □

Corollary 3.36 (Global sections of proper schemes): *Let k be algebraically closed and X proper over k. Then $\Gamma(X, \mathcal{O}_X) = k$.*

Proof. The global sections are precisely given by morphisms of schemes $f : X \to \mathbb{A}^1_k$. We can apply the previous proposition with $Z = X$ and get that $f(X)$ is a closed, proper subscheme of $\mathbb{A}^1_k$ and hence consists of a single point (it cannot be the whole of $\mathbb{A}^1_k$ since it is not proper, by virtue of not being universally closed for instance). But the closed points of $\operatorname{Spec} k[x]$ are precisely in bijection with $\alpha \in k$. □

3.3.4 Projective schemes and the properness of projective morphisms

Definition 3.37 (Projective schemes): *Recall that for rings $A \to B$, we have the following pullback square:*

$$\begin{tikzcd} \mathbb{P}^n_B \arrow[r] \arrow[d] & \mathbb{P}^n_A \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

This motivates the definition $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y$. A projective morphism $X \to Y$ is a map which factors through a closed immersion i:

$$X \xrightarrow{i} \mathbb{P}^n_Y \to Y$$

We are now ready to prove the following important theorem:

Proposition 3.38 (Projective morphisms of noetherian schemes are proper): *Projective morphisms of noetherian schemes are proper*

Proof. Reduce to the case $\mathbb{P}^n_{\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, using stability under base change.. Then, take the point ζ which is the image of $\operatorname{Spec} K$ in $\mathbb{P}^n_{\mathbb{Z}}$. By induction, can assume that ζ is not in any complement of basic opens $\mathbf{V}(x_i)$, since they are isomorphic to $\mathbb{P}^{n-1}_{\mathbb{Z}}$. In other words, we can assume that $\zeta \in \bigcap \mathbf{D}(x_i)$. Hence, the functions x_i/x_j are invertible in the local ring $\mathcal{O}_\zeta$. But we know that $\kappa(\zeta) \subseteq K$, since a K-valued point is given precisely by this data (Lemma 3.31), so we can look at the images of these elements in K, denoted by f_{ij}. These satisfy a cocycle condition. Put $g_i = v(f_{i0})$, where v is the valuation on K, which is integer-valued (it extends the valuation on R). This has a minimal element g_k and hence $v(f_{ik}) = g_i - g_k \geq 0$ and so $f_{ik} \in R$. This allows us to define a homomorphism

$$\varphi : \mathbb{Z}[x_0/x_k, \ldots, x_n/x_k] \to R$$

$$x_i/x_k \mapsto f_{ik}$$

This defines a map $\operatorname{Spec} R \to \mathbf{D}(x_k)$, showing the existence of a lift in the criterion. This morphism is furthermore unique by construction. □

Proposition 3.39 (Products of projective schemes): *Let $S_\bullet, T_\bullet$ be graded rings with $S_0 = A_0 = A$. We define $S \times_A T = \bigoplus S_d \otimes_A T_d$. Then $\operatorname{Proj}(S \times_A T) \simeq \operatorname{Proj} S \times_A \operatorname{Proj} T$ is the fibre product:*

$$\begin{tikzcd} \operatorname{Proj}(S \times_A T) \arrow[r] \arrow[d] & \operatorname{Proj} T \arrow[d] \\ \operatorname{Proj} S \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

Moreover, $\mathcal{O}(1)$ is given by the exterior product of the two line bundles on the separate projective spaces.

Proof. Check affine localy I suppose. □

3.4 Sheaves of modules

Definition 3.40 (Sheaves of modules): *An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ module coherently with the restriction maps.*

Can define all the usual algebraic operations: tensor, hom sheaves; ideal sheaves; locally free sheaves. May have to sheafify!

These have built in functoriality: given $\mathcal{F}$ an $\mathcal{O}_X$-module and $f : X \to Y$, then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module. Moreover, we have a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, which then gives $f_*\mathcal{F}$ an $\mathcal{O}_Y$-module structure.

Conversely, given $\mathcal{G}$ on Y, then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module and $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is an $\mathcal{O}_X$-module, where we equip $\mathcal{O}_X$ with a $f^{-1}\mathcal{O}_Y$-module structure via the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We thus have two adjoint functors, f_* and f^*.

Note that

$$f^*\mathcal{O}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_Y = \mathcal{O}_X$$

i.e. pullback sends structure sheaves to structure sheaves.

Definition 3.41 (Sheaf associated to a module): *Given an A-module M, can produce a sheaf $\tilde{M}$ such that $\tilde{M}(\mathbf{D}_f) = M_f$ and $\tilde{M}_\mathfrak{p} = M_\mathfrak{p}$.*

The global sections of such a sheaf are unsurprisingly given by M. Importantly, for maps between affine schemes we have $f^*(\tilde{M}) \simeq \widetilde{M \otimes_A B}$ and $f_*\tilde{N} = \widetilde{{}_AN}$, for an A-module M and B–module N, where ${}_AN$ is N considered as an A-module using $A \to B$. Hence, this generalizes the induction-restriction adjunction. In fact, the functor from modules to quasicoherent sheaves is fully faithful, and preserves tensor products and direct sums (since these can be checked stalk-locally and localization commutes with both operations).

Definition 3.42 (Quasicoherent sheaf): *A quasicoherent sheaf is a sheaf that is locally associated to a module, in other words there is an affine cover where this is true. It is coherent if all of these modules are finitely generated.*

As with a lot of these definitions, this is local, so in fact every open affine is associated to a module. To prove this we need a lemma.

Lemma 3.43 (Lemma): *Let $\mathcal{F}$ be quasicoherent over an affine $X = \operatorname{Spec} A$. Given $s \in \Gamma(\mathcal{F})$ restricting to 0 on $\mathbf{D}(f)$, then there is an integer n such that $f^n s = 0$. Conversely, if $t \in \mathcal{F}(\mathbf{D}(f))$, then for a large enough n, $f^n t \in \Gamma(\mathcal{F})$.*

Proof. Cover X by open affines where $\mathcal{F}$ is associated to modules, i.e. $V = \operatorname{Spec} B$ where $\mathcal{F}|_V = \tilde{M}$ for a B-module M. Furthermore, cover V by open sets of the form $\mathbf{D}(g_i)$. An inclusion $\mathbf{D}(g_i) \subset$

V is equivalent to a ring homomorphism $B \to A_{g_i}$. If we call this inclusion ι, then $\mathcal{F}|_{\mathbf{D}(g_i)} = \iota^*\mathcal{F}_V = \iota^*\tilde{M} = \widetilde{M \otimes_B A_{g_i}} = \tilde{M}_i$ for some module M_i. By quasicompactness, a finite number of i's are needed.

For the first part, suppose s restricts to 0 on $\mathbf{D}(f)$. It moreover restricts to sections over the $\mathbf{D}(g_i)$ and $\mathbf{D}(f) \cap \mathbf{D}(g_i) = \mathbf{D}(fg_i)$, and here s is zero, so by definition of localization i.e. s_i being 0 in the module $(M_i)_f$, we have that $f^{n_i}s_i = 0$. By quasicompactness, choose $N >> n_i$, hence $f^N s_i = 0$ and since $\mathbf{D}(g_i)$ cover X, we get $f^N s = 0$.

For the second part, given an element $t \in \mathcal{F}(\mathbf{D}(f))$, we restrict for each i to get an element in $\mathcal{F}(\mathbf{D}(fg_i)) = (M_i)_f$. By definition of localization, there is a $t_i \in \mathcal{F}(\mathbf{D}(g_i))$ restricting to $f^n t$ (again, take n big enough). Now, these t_i's glue for the following reason: on intersections $\mathbf{D}(g_ig_j)$, we have sections t_i and t_j agreeing on $\mathbf{D}(fg_ig_j)$, where they are equal to $f^n t$. Hence, $t_i - t_j$ restricts to 0, and by the first part, there must be an m such that $f^m(t_i - t_j) = 0$ on $\mathbf{D}(g_ig_j)$. By taking m big enough, we glue the local sections $f^m t_i$ to a global section s which restricts to $f^{n+m}t$ on $\mathbf{D}(f)$. $\square$

As a corollary, we see that coherence is local, and we can take any affine open cover, given the existence of one such.

Corollary 3.44 (Coherence is local): *If $\mathcal{F}$ is quasicoherent on X, then for every affine open $U = \operatorname{Spec} A$, we have that the sheaf restricts to the associated module of its sections: $\mathcal{F}_U = \widetilde{\Gamma(\mathcal{F}_U)}$. If X is Noetherian, then $\mathcal{F}$ is coherent if and only if on every affine open it restricts to a finitely-generated sheaf-module.*

Proof. We can reduce to the case X affine, since X has a base for its topology on which $\mathcal{F}$ restricts to an associated module, and hence $\mathcal{F}_U$ is quasicoherent. Put $M = \Gamma(\mathcal{F}_U)$. We would like to show $\mathcal{F}_U = \tilde{M}$. Similarly to the adjunction between global sections and Spec for schemes, there is an adjunction between global sections and the associated sheaf of a module. In other words, we get a map of sheaves $\tilde{M} \to \mathcal{F}$ given on the opens $\mathbf{D}_f$ by $m/f^n \mapsto \operatorname{res}_{\mathbf{D}_f}(m)/f^n$. X can be covered by opens $\mathbf{D}(g_i)$ on which $\mathcal{F}$ restricts to a module-sheaf $\tilde{M}_i$ and the previous lemma tells us that $\mathcal{F}(\mathbf{D}(g_i)) = M_{g_i}$ and so $M_i = M_{g_i}$. Thus the map is an isomorphism on this open cover, and hence glues to an isomorphism $\tilde{M} \simeq \mathcal{F}_U$. $\square$

Proposition 3.45 (Equivalence of categories): *For $X = \operatorname{Spec} A$, the functor $M \mapsto \tilde{M}$ gives an equivalence between two categories:*

$$A - \text{modules} \leftrightarrow \text{Quasicoherent}\,\mathcal{O}_X - \text{modules}$$

In fact, this is also an adjunction.

The next proposition is a down-to-earth proof that for quasicoherent sheaves $\mathcal{F}'$ we have $H^1(X, \mathcal{F}') = 0$.

Proposition 3.46 (Quasicoherent sheaves are acyclic over affine schemes): *Let $X = \operatorname{Spec} A$ be affine, $\mathcal{F}'$ quasicoherent and*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

exact. Then the associated sequence on global sections is also exact.

Proof. Let s be a global section of $\mathcal{F}''$. We first look at neighbourhoods $\mathbf{D}(f)$ where $\mathcal{F} \to \mathcal{F}''$ is surjective, in other words s lifts to a section $t \in \mathcal{F}(\mathbf{D}(f))$. We will show that on these bits, $f^N s$ lifts to a global section of $\mathcal{F}$ for some N. Then, we glue all this data using a partition of unity argument.

Cover X by $\mathbf{D}(g_i)$ where s lifts to a section $t_i \in \mathcal{F}(\mathbf{D}(g_i))$. On the overlap $\mathbf{D}(f) \cap \mathbf{D}(g_i)$, we have that both t_i and t lift s, so by exactness, we can identify $t - t_i \in \mathcal{F}'(\mathbf{D}(fg_i))$. By the lemma, for some n, $f^n(t - t_i)$ lifts to a section u_i of $\mathcal{F}'(\mathbf{D}(g_i))$. Pick large enough n and put $t_i' = f^n t_i + u_i$, a section in $\mathcal{F}$. Then t_i' lifts $f^n s$ on $\mathbf{D}(g_i)$, and also t_i' and $f^n t$ agree on $\mathbf{D}(fg_i)$. Thus, t_i' and t_j' lift $f^n s$ on the overlap and agree on $\mathbf{D}(fg_ig_j)$ and so by the lemma, $f^m(t_i' - t_j') = 0$ for big enough m. Then $f^m t_i'$ glue to give a global section of $\mathcal{F}$ which lift $f^{n+m}s$.

Now, cover X by a bunch of these $\mathbf{D}(f_i)$ where $f^N s$ lift to t_i. The ideal $(f_1^n, \ldots, f_k^N) = A$, since $\operatorname{Spec} A = \bigcup \mathbf{D}(f_i)$, and so, using a partition of unity argument, $1 = \sum a_i f_i^N$ and we can put $t = \sum a_i t_i$, which is a global section lifting s. □

In fact, Serre's criterion shows that this holds in the converse: if $H^i(X, \mathcal{F}) = 0$ for all quasicoherent sheaves, then X must be affine!

Proposition 3.47 (Operations on quasicoherent sheaves): *Let $f : X \to Y$ be a morphism of schemes. Then: the kernel, cokernel and image of a morphism of quasicoherent sheaves, as well as extension of quasicoherent sheaves is quasicoherent. $f^*\mathcal{G}$ is quasicoherent on X if $\mathcal{G}$ is quasicoherent on Y. The same holds for coherent sheaves, if X and Y are Noetherian. If X Noetherian, or f is quasicompact and separated, then $f_*\mathcal{F}$ is quasicoherent on Y whenever $\mathcal{F}$ is quasicoherent on X. If $\mathcal{F}$ is quasicoherent and $U \subset X$ is affine, then $\widetilde{\mathcal{F}(U)} \simeq \mathcal{F}_U$.*

Proposition 3.48 (Corollary): *For a closed subscheme $Y \subset X$, the ideal sheaf*

$$\mathcal{I}_Y := \ker(\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Y)$$

is quasicoherent, since closed immersions are separated and $\iota_*\mathcal{O}_Y$ is quasicoherent by the proposition. Importantly, there is a correspondence between closed subschemes and ideal i.e. subsheaves, given by $Y \mapsto \mathcal{I}_Y$ and $\mathcal{I} \mapsto \operatorname{supp}\mathcal{I}$. In particular, the reverse association sends $\mathcal{I}$ to the support Y with sheaf $\mathcal{O}_X/\mathcal{I}$. Applying this to the affine case, we see that any subscheme of $\operatorname{Spec} A$ *corresponds to some quasicoherent sheaf of ideals $\tilde{\mathfrak{a}}$, where $\mathfrak{a} \subset A$ is an ideal, and this gives us the affine scheme* $\operatorname{Spec} A/\mathfrak{a}$.

Let's see more carefully why this is true. If $x \notin Z$, then near x, all functions vanishing on Z become invertible, and so $\mathcal{O}_{X,x} = \mathcal{I}_{Z,x}$. This shows that $\operatorname{Supp}\mathcal{O}_X/\mathcal{I}_Z \subset Z$. Conversely, if $\mathcal{O}_{X,x} = \mathcal{I}_{Z,x}$, then since Z locally looks like $\mathbb{V}(I)$, we can put $x = \rho \in \operatorname{Spec} A$ and get $A_\rho = I_\rho$. If this were the case, then for all $t \notin \rho, 1/t \in I_\rho$ and hence there is some $i \in I$ such that $t''(i - tt') = 0$, where all the t's are not in ρ. This implies that i cannot be in ρ, which is prime, so $\rho \notin \mathbb{V}(I)$, which is Z locally. This shows the other side of the inclusion, so $Z = \operatorname{Supp}\mathcal{O}_X/\mathcal{I}_Z$.

The other isomorphism goes as follows: taking $\mathcal{I}$ on X, which on local bits is given by $\tilde{I}$, the same argument as above tells us that on those affine bits we get a closed subscheme $\mathbb{V}(I) = \operatorname{Spec} A/I$. This then associates an ideal sheaf which is locally the kernel of the morphism $\mathcal{O}_{\operatorname{Spec} A} \to \iota_*\mathcal{O}_{\operatorname{Spec} A/I}$ which we directly verify is again the quasicoherent sheaf $\mathcal{I}$ given locally by I.

Proposition 3.49 (Annihilators and supports): *Define* $\operatorname{Supp}(s) = \{x \in X | s_x \neq 0\}$. *Then if $X = \operatorname{Spec} A$ and M is an A-module, $\mathcal{F} = \tilde{M}$ is a quasicoherent sheaf, we have* $\operatorname{Supp}(m) = \mathbb{V}(\operatorname{Ann}(m))$. *Moreover, when A is Noetherian, M is finitely generated and* $\operatorname{Supp}(\mathcal{F}) = \{x | \mathcal{F}_x \neq 0\} = \mathbf{V}(\operatorname{Ann}(m))$. *In particular, when $M = I$ is an ideal, we recover that* $\operatorname{Supp}(\mathcal{O}_X/\tilde{I}) = \mathbf{V}(\operatorname{Ann}(A/I)) = \mathbf{V}(I)$.

Proof. For the first, the complement of the support of m consists of all primes ρ such that $m_\rho = 0 \in M_\rho$. This means that $m = 0$ near ρ as well, i.e. in some basic open D_f with $f \notin \rho$. In other words, $m = 0 \in A[1/f]$ and hence by definition, $f^N m = 0 \in A$. But since $f \notin \rho$ and ρ is prime, $f^N \notin \rho$ and hence $f^N \in \operatorname{Ann}(m)$ but not in ρ. Therefore, ρ is in the complement of $\mathbf{V}(\operatorname{Ann}(m))$. The converse follows the same steps in reverse, and by taking complements we get what we want.

For the second part, suppose m_i generate M. Then $M_\rho = 0 \iff (m_i)_\rho = 0$ for all i and hence

$$\begin{aligned}\operatorname{Supp}(\mathcal{F})^c = \{\rho | M_\rho = 0\} = \bigcap \{\rho | (m_i)_\rho = 0\} = \bigcap \operatorname{Supp}(m_i)^c = \\ = \bigcap \mathbb{V}(\operatorname{Ann}(m_i))^c = \bigcup \mathbb{V}(\operatorname{Ann}(m_i)) = \mathbb{V}(\operatorname{Ann}(M))\end{aligned}$$

□

Proposition 3.50 (Projection formula): *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, $\mathcal{F}$ an $(X, \mathcal{O}_X)$-module and $\mathcal{E}$ a locally free $(Y, \mathcal{O}_Y)$-module of finite rank. Then*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \simeq f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$$

Looking locally, we also get the fact that the support of a coherent sheaf on a Noetherian scheme is closed!

Proof. We create first create a map using the adjunction and then verify locally that it is an isomorphism. Firstly, the identity map $f_*\mathcal{F} \to f_*\mathcal{F}$ creates a map $f^*f_*\mathcal{F} \to \mathcal{F}$ via the adjunction. Then, we get

$$f^*(f_*\mathcal{F} \otimes \mathcal{E}) \simeq f^*f_*\mathcal{F} \otimes f^*\mathcal{E} \to \mathcal{F} \otimes f^*\mathcal{E}$$

since f^* respects tensor products. Finally, we apply the adjunction again to get a morphism

$$f_*\mathcal{F} \otimes \mathcal{E} \to f_*(\mathcal{F} \otimes f^*\mathcal{E})$$

which can be verified to be an isomorphism locally, by taking $\mathcal{E} = \mathcal{O}_Y$ (the rank doesn't matter) and using the fact that f^* sends structure sheaves to structure sheaves. □

3.4.1 Sheaves of modules on projective space and line bundles

For a graded module M we can again associate a sheaf defined by

$$(\tilde{M})(\mathbf{D}(f)) = M[1/f]_0$$

Hence, $\tilde{M}|_{\mathbf{D}_f} = \widetilde{M[1/f]_0}$

For example, we can define $\mathcal{O}(d)$ to be the sheaf we get from applying Proj to the same graded ring, but with shifted grading, which is a graded $A[x_0, \ldots, x_n]$ module. Locally, $\mathcal{O}_X(d)$ for $X =$ Proj S looks like the degree d elements in $S[1/f]$ which, if $f \in S_1$, is isomorphic to $\mathcal{O}_X$ locally via multiplication by f^d. Hence, this is a locally free sheaf of rank 1.

Example : For $\mathbb{P}^1$, we can take an open cover $\mathbf{D}(x_0), \mathbf{D}(x_1)$. In the first, we have sections of the form $x_1, x_1^2/x_0, \ldots$, and on the second we have things like $x_0, x_0^2/x_1, \ldots$. To glue, we cannot have denominators, so we get the linear homogenous polynomials. More generally,

$$\mathcal{O}(d)(\mathbb{P}^1) = \mathrm{degree\,d\,homogenous\,polynomials}$$

Definition 3.51 (Associated graded module of a sheaf of modules): *Given $X =$ Proj S and $\mathcal{F}$ a sheaf of modules, we define a graded S-module via $\Gamma_*(\mathcal{F}) = \oplus\Gamma(X, \mathcal{F}(n))$, where $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.*

Proposition 3.52 (Coherence is local, v2): *For a graded ring S which is generated by S_1 as an algebra over S_0. If $X = \operatorname{Proj} S$ and $\mathcal{F}$ is a quasicoherent sheaf on X, then $\mathcal{F} \simeq \widetilde{\Gamma_*(\mathcal{F})}$. Moreover, we still have a correspondence of ideal sheaves given by $I \subset A[x_0, ..., x_n]$ and closed subschemes of $\mathbb{P}^n_A$. A scheme over $\operatorname{Spec} A$ is projective if and only if it is $\operatorname{Proj} S$ where $S_0 = A$ and S is f.g. over S_0 by S_1.*

Definition 3.53 (Ample line bundles): *$\mathcal{L}$ is a basepoint-free line bundle on X is it arises by pulling back the twisted sheaf $\mathcal{O}(1)$ along $f : X \to \mathbb{P}^n$. $\mathcal{L}$ is very ample if f can be taken to be a closed immersion. It is ample if $\mathcal{L}^{\otimes n}$ is very ample for some n.*

3.5 Relative Spec and Proj

Given an A-algebra B, we can associate to it $\operatorname{Spec} B \to \operatorname{Spec} A$. This satisfies the universal property that, since Spec and global section are adjoint:

$$\operatorname{Hom}_{\text{Schemes over A}}(W, \operatorname{Spec} B) \simeq \operatorname{Hom}_A(B, \Gamma(W))$$

This corresponds to taking global sections in the following diagram:

$$\begin{tikzcd} W \arrow[rr] \arrow[dr] & & \operatorname{Spec} B \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd} \qquad \begin{tikzcd} \Gamma(W) & & B \arrow[ll] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We want to globalize this construction, in the sense that if we're given a quasicoherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$, we associate to it a universal object $\mathcal{S}pec B$. We would like that on affine patches where X is given by $\operatorname{Spec} A$ and $\mathcal{B} = \tilde{B}$ for some A-module B, then $\mathcal{S}pec\,\mathcal{B} = \operatorname{Spec} B$ over $\operatorname{Spec} A$. The universal property is the following:

Definition 3.54 (Universal property of global Spec): *For any map of X-schemes there corresponds a map of $\mathcal{O}_X$-algebras:*

$$\begin{tikzcd} W \arrow[rr, "\alpha"] \arrow[dr, "\mu"'] & & \mathcal{S}pec\,\mathcal{B} \arrow[dl, "\beta"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mu_*\mathcal{O}_W & & \mathcal{B} \arrow[ll] \\ & \mathcal{O}_X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Put differently,

$$\operatorname{Hom}_X(W, \mathcal{S}pec\,\mathcal{B}) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{B}, \mu_*\mathcal{O}_W)$$

Now, by Yoneda nonsense or by considering affine opens, we can conclude that $\mathcal{B} \simeq \beta_*\mathcal{S}pec\,\mathcal{B}$. This follows, since the diagram on the left induces the following diagram on sheaves:

$$\begin{tikzcd} \mu_*\mathcal{O}_W & & \beta_*\mathcal{S}pec\mathcal{B} \arrow[ll] \\ & \mathcal{O}_X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The maps are $\beta^\#, \mu^\#, \beta_*\alpha^\#$. We can see that on affine opens they both give the same answer.

The idea is that on affine bits of X given by $U = \operatorname{Spec} A$, then $\mathcal{A}(U)$ is an A-algebra of which we can take Spec. Global Spec glues these together:

$$\begin{tikzcd} \operatorname{Spec}\mathcal{A}(U) \arrow[r] \arrow[d, "\pi_U"'] & \underline{\operatorname{Spec}}\,\mathcal{A} \arrow[d, "\pi"] \\ U \arrow[r] & X \end{tikzcd}$$

These local schemes glue. If $p \in U$, with residue field $\kappa(p)$, we thus have the fiber $\pi_U^{-1}(p)$ by the Cartesian diagram:

$$\begin{tikzcd} \operatorname{Spec}(\mathcal{A}(U) \otimes \kappa(p)) \arrow[r] \arrow[d] & \operatorname{Spec}\mathcal{A}(U) \arrow[d] \\ \operatorname{Spec}\kappa(p) \arrow[r] & U \end{tikzcd}$$

The point is that p belongs to many U's, so the fiber in the total Global Spec should correspond to taking a limit over all U containing p. One can draw a big diagram and see that there are coherent maps $\pi^{-1}(p) \to \pi_U^{-1}(p)$, which should build to an isomorphism

$$\pi^{-1}(p) \simeq \varprojlim \pi_U^{-1}(p) = \varprojlim_{p\in U} \operatorname{Spec}(\mathcal{A}(U) \otimes \kappa(p)) =$$

$$= \operatorname{Spec}(\varinjlim_{p\in U} \mathcal{A}(U) \otimes \kappa(p)) = \operatorname{Spec}(\mathcal{A}_p \otimes \kappa(p))$$

(Spec sends limits to colimits as it is right adjoint... or sth.)

I suppose $\mathbb{A}^n_S = \underline{\operatorname{Spec}}\,\mathcal{O}_S[x_1,\dots,x_n]$. Then we can pull back along the structure map $f : X \to S$ to get a quasicoherent sheaf of $\mathcal{O}_X$-algebras on X given by $f^*\mathcal{O}_S[x_1,\dots,x_n]$. The global Spec of this should fit into a diagram:

$$\begin{tikzcd} \underline{\operatorname{Spec}}\, f^*\mathcal{O}_S[x_1,\dots,x_n] \arrow[r] \arrow[d] & X \arrow[d] \\ \underline{\operatorname{Spec}}\,\mathcal{O}_S[x_1,\dots,x_n] \arrow[r] & S \end{tikzcd}$$

showing that this sheaf of algebras produces $X \times_S \mathbb{A}^n_S$. But $f^*\mathcal{O}_S = \mathcal{O}_X$, consistent with the fact that we have a Cartesian diagram:

$$\begin{tikzcd} X \times_S \mathbb{A}^n_S \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{A}^n_S \arrow[r] \arrow[d] & S \arrow[d] \\ \mathbb{A}^n_{\mathbb{Z}} \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

Example (Blowups): See https://math.mit.edu/~mckernan/Teaching/09-10/Spring/18.726/l_10.pdf for more. Take $A = k[x_1,\dots,x_n]$ producing the affine scheme $\mathbb{A}^n_k$. We have an ideal $I = (x_0,\dots,x_n)$ corresponding to the point 0. Thiis creates a graded ideal

$$\bigoplus_{d\geq 0} I^d$$

which we can think of as a quasicoherent sheaf of algebras on $\mathbb{A}^n_k$. We can take global Proj of this, which is just normal Proj, since we're working over an affine scheme, to get the blowup:

$$\mathrm{Bl}_I\,\mathbb{A}^n_k = \mathrm{Proj}\,(\bigoplus_{d\geq 0} I^d)$$

Now, note that we have a surjective map

$$A[y_1,\dots,y_n] \xrightarrow{y_i\mapsto x_i} \bigoplus_{d\geq 0} I^d$$

which has kernel generated by $(x_iy_j - x_jy_i)$. The induced map on Proj is a closed immersion

$$\mathrm{Bl}_I\mathbb{A}^n_k \to \mathbb{P}^{n-1}_A$$

which realizes the blowup as the subscheme given by the vanishing locus of the homogenous polynomials $(x_iy_j - x_jy_i)$, which are the usual equations of the blowup.

3.6 Divisors

3.6.1 Weil divisors

Assume X is integral, Noetherian, separated and regular in codimension 1, i.e. A_p is a DVR for height 1 primes p. Examples are affine and projective space.

Recall that the dimension of X is the length of the longest chain of nonempty irreducible closed subsets $Z_0 \subset Z_1 \subset ... \subset X$. Can define codimension similarly.

Definition 3.55 (Prime and Weil divisors): *A prime divisor is a closed integral subscheme of codimension* 1*. A Weil divisor is a formal sum of such prime divisors:*

$$W = \sum n_D[D]$$

It is effective if $n_D \geq 0$ for all D.

Note that since X is integral, it has a unique generic point η: see the section on irreducible components. Put $\kappa(X) = \mathcal{O}_{X,\eta}$ the function field of X which contains all of the rings of sections and local rings. Given $f \in \kappa(X)^\times$, we define:

Definition 3.56 (Principal divisors):

$$\mathrm{div}(f) = \sum_{Y \subset X, \text{prime}} n_Y(f)[Y]$$

$n_Y(f)$ is the valuation of f in the DVR $\mathcal{O}_{Y,\eta_Y}$

The local ring is a DVR with fraction field $\kappa(X)$ since we are assuming Y is integral of codimension 1. The point is that any open containing η_Y actually contains η as well, since it specializes to it, being the generic point, so this valuation makes sense, by looking at direct limits.

Compare with the complex setting: if Y is cut out by g, then the local ring has a maximal ideal generated by g and the valuation of a meromorphic f is the power of g that divides it. Note that $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$. The sum is finite for the following reason:

Proposition 3.57 (Well-definedness): *$n_Y(f) = 0$ for all but finitely many Y*

Proof. Find an affine open $U = \operatorname{Spec} A$ where f, a rational function on X, is regular. In other words, $f \in A \subset \kappa(X)$. Then $X - U$ is a closed proper subscheme of the Noetherian X, so contains only finitely many prime divisors. On U with Y meeting it, $n_Y(f) \geq 0$ and is greater than 0 precisley when Y is contained in the proper closed subset defined by the ideal Af i.e. $\mathbb{V}(f)$, which contains finitely many closed irreducible subsets of codimension one. □

Definition 3.58 (Class group): *The class group of X is defined to be the Weil divisors modulo the principal divisors:*

$$\mathrm{Cl}(X) := \frac{\mathrm{Div}(X)}{\mathrm{Prin(X)}}$$

Fact: For A noetherian, A is a UFD if and only if $\mathrm{Cl}\,\mathrm{Spec} A = 0$ and $\mathrm{Spec}\, A$ is normal. In particular, the class group of affine space is trivial.

Proposition 3.59 (Class group of projective space): *For $X = \mathbb{P}^n$, there is a map $\mathrm{Div}(X) \to \mathbb{Z}$ given by $\sum n_i Y_i \mapsto \sum n_i \deg Y_i$ which is zero on principal divisors and induces an isomorphism $\mathrm{Cl}(X) \simeq \mathbb{Z}$, generated by the hypersurface $H = \{x_0 = 0\}$.*

Proof. The map is well defined, since $\deg \mathrm{div}(f) = 0$. This is since the function field is given by rational functions of total degree 0 - just look at an affine open, where the functinos are given by $\mathbb{Z}[x_0, ..., x_n][1/x_i]_0$ and then take the field of fractions, which is precisely the rational functions of total degree 0. Moreover, it is surjective, by looking at the image of H. Injectivity follows, since if

$$\sum n_i (\deg Y_i) = 0$$

with $Y_i = \mathbb{V}(g_i), \deg(g_i) = \deg Y_i$, then can put $f = \prod g_i^{n_i}$ producing a principal divisor. □

Proposition 3.60 (Excision sequence): *Let $Z \subset X$ be a proper closed subset of X, $U = X - Z$. We have a restriction map*

$$\mathrm{Cl}(X) \to \mathrm{Cl}(U), D \mapsto D \cap U$$

When the codimension of Z in X is greater than 1, this is an isomorphism. If the codimension is equal to 1 and Z is irreducible, there is an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$$

where the kernel is generated by the image of Z.

Example (Computations of class groups): Immediately, we see by the excision sequence that if Y is a hypersurface of degree d in $\mathbb{P}^n$, then $\mathrm{Cl}(\mathbb{P}^n - Y) \simeq \mathbb{Z}/d\mathbb{Z}$. As a second example, take the affine variety $A = k[x,y,z]/(xy - z^2)$ with spectrum X. Consider the subscheme Z given by the vanishing of x, z. Thinking inside X, this is cut out by a single equation $Z = \mathbb{V}_X(x)$, since $x = 0$ implies $z^2 = 0$. Hence, it is of codimension 1 in X (and codimension 2 in affine three-space). We see that $X - Z = \mathbf{D}_X(x) = \mathrm{Spec}\, k[x,y,z]/(xy - z^2)[1/x] \simeq k[x, x^{-1}, z]$ which is a UFD, so the class group vanishes. We also see that the divisor of x is given by

$$\mathrm{div}(x) = v_Z(x)Z$$

where v_Z is the valuation of x in the DVR $\mathcal{O}_{X,\eta_Z}$. This local ring is the localization at the prime ideal $(xy - z^2, x, z)$, if we think inside affine space, and hence in it, we have inverted y and $x = \frac{1}{y}z^2$ has valuation 2, since z generates the local ring. Therefore, $2Z$ is principal. We only need to verify that Z is not principal, from which it will follow that $\mathrm{Cl}(X) \simeq \mathbb{Z}/2\mathbb{Z}$ by the excision sequence. To verify it is not principal, we put $\rho = (x,z) \subset \mathfrak{m} = (x,y,z)$. Then $\mathfrak{m}/\mathfrak{m}^2$ is a three dimensional vector space over k generated by the images of x, y, z and the image of ρ contains y, z, hence cannot be principal.

Theorem 3.61 (Class group of product with affine space):

$$\mathrm{Cl}(X \times \mathbb{A}^1) \simeq \mathrm{Cl}(X)$$

Proposition 3.62 (Class group of product with projective space):

$$\mathrm{Cl}(X \times \mathbb{P}^n) \simeq \mathrm{Cl}(X) \oplus \mathbb{Z}$$

Proof. Use excision for the closed set $Z = X \times \mathbb{V}(x_0)$ to get

$$\mathbb{Z} \to \mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}(X \times \mathbb{A}^n) \simeq \mathrm{Cl}(X) \to 0$$

The right map has a splitting given by the projection $\pi^* : \mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{A}^n)$. We need to show that the left map is injective. We have a commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}\{Z\} \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Cl}(X \times \mathbb{P}^n) \arrow[d, "\iota^*"] \\
\mathbb{Z}\{H\} \arrow[r, "\simeq"'] & \mathrm{Cl}(\mathbb{P}^n)
\end{tikzcd}$$

This provides a retraction for the map, showing it is injective. □

Remark: On projective subvarieties, there is a well-defined restriction map $\mathrm{Cl}(\mathbb{P}^n) \to \mathrm{Cl}(V)$ which is moreover injective.

Theorem 3.63 (Class group of affine scheme): *If A is a noetherian domain, then A is a UFD precisely when* $\operatorname{Spec} A$ *is normal (i.e. A is integrally closed) and* $\mathrm{Cl}(\operatorname{Spec} A) = 0$.

Example (Quadric hypersurfaces): Let $A = \operatorname{Spec} k[x_0, \dots, x_n]/(x_0^2 + \dots + x_n^2)$, k has characteristic not equal to 2. Then, when $n > 1$, this ring is integrally closed, as we can rewrite it as $k[x_1, \dots, x_n][x_0]/(x_0^2 + x_1^2 + \dots + x_n^2)$. But now, $f = x_1^2 + \dots + x_n^2$ is square free as an element of $k[x_1, \dots, x_n]$ and we can just appeal to the fact that $k[x_1, \dots, x_n][z]/(z^2 - f)$ is integrally closed for such situations - this is done by looking at the field of fractions, which is Galois of order 2 over $k(x_1, \dots, x_n)$ and $h+gz$ has minimal polynomial $X^2 - 2gX + (g^2 - h^2 f)$, whose coefficients lie in $k[x_1, \dots, x_n]$ precisely when $g, h \in l[x_1, \dots, x_n]$ since we assume the characteristic is not 2.
For $n = 2$, we can rewrite the equation as $y_0 y_1 = y_2^2$ for $y_0 = x_0 - ix_1, y_1 = x_0 + ix_1$ and this is the example from 3.6.1, so the class group is $\mathbb{Z}/2\mathbb{Z}$.
For $n > 3$, we can consider excision for the closed set $Z = \mathbb{V}_X(x_0)$:

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X - Z) \to 0$$

The complement $X - Z$ has coordinate ring with x_0 inverted, hence is $k[x_0^{\pm}, x_1, \dots, x_n]/(x_0 x_1 = x_2^2 + \dots + x_n^2) \simeq k[x_0^{\pm}, x_2, \dots, x_n]$ by eliminating x_1, and this is a UFD. It thus has vanishing class group. We would like to show that $\mathrm{Cl}(X) = 0$ and this will be done by showing that the Z is principal i.e. $Z = \operatorname{div}(x_0)$. All that we need is to find the valuation of x_0 in the local ring at the generic point of Z. This is the localization at $(x_0, x_0 x_1 - x_2^2 - \dots - x_n^2)$. In it x_1 is a unit and $x_2^2 + \dots + x_r^2$ is a generator, as is x_0, which follows from the fact that $x_2^2 + \dots + x_n^2$ is irreducible in $k[x_2, \dots, x_n]$ for $n > 3$ (this follows by just expanding out any linear product and comparing coefficients). In other words, we are dealing with a localization $k[x_2, \dots, x_n]_{(x_2^2 + \dots + x_n^2)}$ where the principal ideal we are localizing at is prime and hence generates the maximal ideal. All in all, in this case the valuation is 1 and Z is principal. As a corollary, we get that $k[x_0, \dots, x_n]/(x_0^2 + \dots + x_n^2)$ is a UFD for $n > 3$.
The remaning case $n = 3$ is the affine hypersurface given by the equation $x_0 x_1 = x_2 x_3$. This is the affine cone X lying above the projective quadric Q, which is just $\mathbb{P}^1 \times \mathbb{P}^1$ with class group $\mathbb{Z} \oplus \mathbb{Z}$. There is, however, an excision sequence for situations like this (Harthosrne exercise 6.6.2):

$$0 \to \mathbb{Z} \to\to \mathrm{Cl}(Q) \to \mathrm{Cl}(A) \to 0$$

The first map sends $1 \mapsto Q \cdot H = (1,1)$ and hence $\mathrm{Cl}(X) = \mathbb{Z}$.

Definition 3.64 (Sheaf associated to a Weil divisor):

$$\mathcal{O}_X(D)(U) := \{t \in K(X)^\times : \operatorname{div}_U t + D_U \geq 0\} \cup \{0\}$$

In other words, the rational functions on U where we allow poles of the order of a positive coefficient in D and demand zeros of order negative coefficients.

Example (Hypersurface in projective space): H in $\mathbb{P}^n$ produces a sheaf whose sections are rational functions allowing a pole at $t = 0$. By multiplying with t, we get a homogenous degree 1 polynomial, i.e. a section of $\mathcal{O}(1)$. In fact, the two are the same!

3.6.2 Cartier divisors

Firstly, we define the sheaves of rational and invertible functions on a scheme X.

Definition 3.65 (Sheaves of rational functions): *For an affine open $U = \operatorname{Spec} A$ in X, we can associate*

$$U \mapsto S^{-1}A,$$

where S is the multiplicative subset of nonzero divisors. Sheafifying, this gives us the additive sheaf $\mathcal{K}$ of rational functions. By taking only the nonzero elements, we get another sheaf $\mathcal{K}^\times$, but this time the groups have a multiplicative structure. Finally, sheafifying

$$U \mapsto A^\times$$

gives us the sheaf $\mathcal{O}_X^\times$.

Definition 3.66 (Cartier divisors): *A Cartier divisor is a global section of the quotient sheaf*

$$\mathcal{K}^\times / \mathcal{O}_X^\times$$

It is effective if the f_i defined below can be taken to be in $\Gamma(U_i, \mathcal{O}_{U_i})$. An effective Cartier divisor defines a sheaf of ideals locally generated by the f_i which corresponds to a subscheme of codimension 1.

Practically, we are given rational function f_i on an open cover U_i such that they agree, mod $\mathcal{O}_X^\times$, on the overlaps, i.e. $f_i/f_j \in \mathcal{O}_X^\times$. For exampe, when $X = \operatorname{Spec} k[x_0, \ldots, x_n]$, a Cartier divisor is a ratio of polynomials up to scaling. In this case, this coincides with the Weil divisors, but this is not true in general.

Given a Cartier divisor $\mathcal{D}$ given by such data, we can get a Weil divisor by the following rule: given $Y \subset X$ codimension 1, integral, find a U_i containing the generic point η_Y. Now put $n_Y(\mathcal{D}) = v_Y(f_i)$. This is independent of the choices: since $f_i/f_j \in \mathcal{O})_X^\times(U_i \cap U_j)$, we have that $v_Y(f_i/f_j) =$

$v_Y(f_i) - v_Y(f_j) = 0$.

Proposition 3.67 (Cartier and Weil divisors sometime coincide): *When X is Noetherian, integral, separated and all its local rings are UFD's, then the association $\mathcal{D} \mapsto \sum n_Y(\mathcal{D})Y$ respects principal divisors and is a bijection between the Cartier and Weil divisors.*

Proof. We give a converse construction: start with a Weil divisor D. This induces Weil divisors D_x on the local schemes $\operatorname{Spec}\mathcal{O}_x$. By the Fact, the class group of a UFD is trivial, so D_x is principal, $D_x = \operatorname{div} f_x$, $f_x \in K$ (note that $\mathcal{K}$ is the constant sheaf on the function field K of X, as X is integral). Hence, D and $\operatorname{div} f_x$ restrict to the same thing on the local scheme, so they must differ only at prime divisors not passing through x. But there are only finitely many of these occuring in D or $\operatorname{div} f_x$, hence they must actually agree on an open U_x. We can now cover X with such opens, and the f_x's produce a Cartier divisor □

Note: the short exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}_X^* \to 0$$

Using this, we get a LES

$$0 \to H^0(X, \mathcal{O}_X^*) \to H^0(X, \mathcal{K}^*) \to H^0(\mathcal{K}^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \simeq \operatorname{Pic}(X)$$

This is a map from the Cartier divisors to Pic(X) whose kernel consists of the principal divisors.

3.6.3 The Picard group

Definition 3.68 (Picard group): *The group of invertible (locally free rank 1) sheaves is denoted* $\operatorname{Pic} X$ *and is isomorphic to $H^1(X, \mathcal{O}_X^\times)$, by associating a line bundle its cocycle.*

Given a Cartier divisor given by a system $\{f_i, U_i\}$, we can create a subsheaf of $\mathcal{K}$ by taking $\mathcal{L}(D)(U_i)$ to be the submodule generated by f_i^{-1}. An example is $X = \mathbb{P}^n$ and $D = H$, whence $\mathcal{L}(D)$ gives us the linear homogenous polynomials.

Proposition 3.69 (Cartier divisors and the Picard group): *The association $D \mapsto \mathcal{L}(D)$ gives a 1-1 correspondence between Cartier divisors and invertible sheaves such that $\mathcal{L}(D_1 - D_2) \simeq \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$ and $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \simeq \mathcal{L}(D_1)$. This gives an injective homomorphism from the group of Cartier divisors modulo principal Cartier divisors* CaCl *into the Picard group, which is an isomorphism when X is integral.*

As a corollary, we get:

Proposition 3.70 (Corollary): *When X is Noetherian, integral, irreducible and separated, then*

$$\operatorname{Cl} X \simeq \operatorname{CaCl} X \simeq \operatorname{Pic} X$$

In particular, for $X = \mathbb{P}^n$, we have that the class group is $\mathbb{Z}$, generated by the hypersurface corresponding to the invertible sheaf $\mathcal{O}(1)$, implying that any line bundle over projective space is of the form $\mathcal{O}(l)$.

Proposition 3.71 (Ideal sheaves and Cartier sheaves): *Given an effective cartier divisor D with associated locally principal closed subscheme (i.e. associated to the sheaf of ideals that D generates) Y, we have*

$$\mathcal{I}_Y \simeq \mathcal{L}(-D)$$

Proof. Recall that $\mathcal{L}(D)$ is locally generated by f_i^{-1} and hence $\mathcal{L}(-D)$ is locally generated by f_i , which is precisely the ideal sheaf generated by D. $\square$

Hence, the ideal sheaf of D and its associated divisor sheaf are dual.

Example (Example): we saw that the associated Weil divisor sheaf of the prime divisor of a hyperplane H in projective space produces $\mathcal{O}(1)$. On the other hand, using this definition, we see that $\mathcal{L}(-H) \simeq \mathcal{I}_H \simeq \mathcal{O}(-1)$.

3.6.4 Summary of the relation between Weil and Cartier divisors and invertible sheaves

We saw how to get from Cartier divisors given by a system $\mathcal{D} = \{U_i, f_i\}$ to line bundles $\mathcal{L}_{\mathcal{D}}$ whose value on U_i is $\frac{1}{f_i}\mathcal{O}_X(U_i)$. Moreover, we can also associate a Weil divisor $D = \sum_Y v_Y(f_i)[Y]$, where i is such that $\eta_Y \in U_i$. In favorable cases, there is a more complicated inverse association, given by the dotted line. Finally, given a Weil divisor D, we also can associate to it a sheaf $\mathcal{O}_X(D)$ whose values on open sets are rational functions constrained by D. We get the following diagram, which we show is commutative:

$$\begin{tikzcd}
\text{Cartier} \arrow[rr] \arrow[dr] & & \text{Weil} \arrow[ll, dotted, bend right] \arrow[dl] \\
& \text{Line bundles (invertible sheaves)} &
\end{tikzcd}$$

We compare:

$$\begin{tikzcd}
\mathcal{D} = \{U_i, f_i\} \arrow[rr] \arrow[dr] & & D = \sum v_Y(f_i)[Y] \arrow[dl] \\
& \mathcal{L}_{\mathcal{D}}(U_j) = \frac{1}{f_j}\mathcal{O}_x(U_j) \overset{?}{\Longrightarrow} \mathcal{O}_X(D)(U_j) &
\end{tikzcd}$$

By definition,

$$\mathcal{O}_X(D)(U_j) = \{t \in K(X)^\times | \operatorname{div}|_{U_j} t + D|_{U_j} \geq 0\}$$

In other words, we are seeking rational functions constrained by D. However, on U_j, for any Y such that $Y \cap U_j$ is nonempty, we can just take $i = j$ in the formula for the D, hence the restriction becomes:

$$D|_{U_j} = \sum v_Y(f_j)[Y \cap U_j] = \operatorname{div}(f_j)$$

Hence, on these opens, the restriction of D is principal and the condition becomes:

$$\operatorname{div}(t) + \operatorname{div}(f_j) = \operatorname{div}(f_j t) \geq 0 \iff t \in \frac{1}{f_j}\mathcal{O}_X(U_j) = \mathcal{L}_{\mathcal{D}}(U_j)$$

We have shown these sheaves agree on an open cover of X.

Proposition 3.72 (Line bundles and sections): *Given a nonzero $s \in H^0(X, L)$, it gives us a zero-set hypersurface $Z(s)$ whose associated line bundle recovers L:*

$$\mathcal{O}(Z(s)) \simeq L$$

3.7 Line bundles and projective space as a moduli space

3.8 Sheaf cohomology

We study the derived functors of the global sections functor $\Gamma(X, \mathcal{F})$. They are functorial: given $f : X \to Y$, we get an induced map

$$f^* : H^i(Y, \mathcal{F}) \to H^i(X, f^{-1}\mathcal{F})$$

Moreover, for the constant sheaf $\underline{\mathbb{Z}}$, this cohomology will agree with the usual Betti cohomology. As with any derived functor, it turns SES's into LES's.

$$\begin{tikzcd}
H^i(X; \mathcal{F}') \arrow[r] & H^i(X; \mathcal{F}) \arrow[r] & H^i(X; \mathcal{F}'') \arrow[dll] \\
H^{i+1}(X; \mathcal{F}') \arrow[r] & H^{i+1}(X; \mathcal{F}) \arrow[r] & H^{i+1}(X; \mathcal{F}'')
\end{tikzcd}$$

To compute sheaf cohomology, one can find an injective resolution of it:

$$0 \to \mathcal{F} \to I^1 \to \ldots$$

Then, by replacing $\mathcal{F}$ by the complex $I_\bullet$ we calculate the homology of the complex obtained by taking global sections:

$$H^i(X,\mathcal{F}) = \frac{\ker}{\mathrm{im}} \text{ of the } i \text{ -th bit in } \Gamma(I_\bullet)$$

As usual, can replace the injectives by any acyclic resolution, examples of which are flasque, soft and fine sheaves.

Theorem 3.73 (Grothendieck vanishing): *If X is a Noetherian scheme of dimension n, then $H^i(X,\mathcal{F}) = 0$ for $i > n$.*

3.8.1 Cech cohomology

A more convenient way to calculate sheaf cohomology is using Cech cohomology. Given a "nice" cover U_i of X, we can consider the complex of Cech p-cochains:

$$C^p(X) = \prod_{i_0<\ldots<i_p} \mathcal{F}(U_{i_0\ldots i_p})$$

$$d : C^p \to C^{p+1}$$

$$(d\alpha)_{i_0\ldots i_{p+1}} = \sum_{j=0}^{k+1} (-1)^j \operatorname{res} \alpha_{i_0\ldots\hat{i_j}\ldots i_{p+1}}$$

In other words, we are taking a section in all p–fold intersections and then the differential, evaluated on some $p+1$-fold intersection, is going to be given by an alternating sum of the restrictions of α on the different bits. By the usual combinatorial argument, $d^2 = 0$.

In the case of schemes, if all U_i are affine, as well as their intersections and $\mathcal{F}$ is quasicoherent, then this will compute the sheaf cohomology! In particular, if X is separated, intersections of affines are affine, so we only need the U_i to be affine.

3.8.2 Cohomology of projective space

We will calculate the cohomology of projective space. Let

$$\mathcal{F} = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}(d)$$

be the sum of all line bundles on $\mathbb{P}^n$. Recall that the global sections of $\mathcal{O}(d)$ was given by degree d homogenous polynomials. Hence, $H^0(X,\mathcal{F}) = k[x_0,\ldots,x_n]$.

Take the affine open cover $U_i = \mathbf{D}(x_i) = \operatorname{Spec} k[x_0,\ldots,x_n][1/x_i]_0$. The p-fold intersection $U_{i_0\ldots i_p}$ will then correspond to just localizing at all the x_{i_k}. Hence,

$$\mathcal{F}(U_{i_0\ldots i_p}) = k[x_0,\ldots,x_n]_{x_{i_0}\ldots x_{i_p}}$$

Hence,

$$C^{n-1} = \bigoplus k[x_0,...,x_n]_{x_0...\widehat{x_k}...x_n}$$
$$C^n = \bigoplus k[x_0,...,x_n]_{x_0...x_n}$$

To calculate H^n we need to calculate the image of $d : C^{n-1} \to C^n$. But this is taking the k span of all monomials with integer powers modulo those where at least one of the monomials has a nonnegative power. The resulting thing is going to be spanned over k by all monomials where all the powers are negative, hence

$$H^n(\mathbb{P}^n, \mathcal{F}) \simeq \frac{1}{x_0...x_n} k[x_0^{-1},...,x_n^{-1}]$$

For the other degrees $0 < r < n$, we claim that the cohomology is 0 and we proceed by induction. We can embed $\mathbb{P}^{n-1} \to \mathbb{P}^n$ as the vanishing set of x_0 and get an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\times x_0} \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_{\mathbb{P}^{n-1}} \to 0$$

This is exact as the multiplication my x_0 map lands in the ideal sheaf $\mathcal{I}_{\mathbb{P}^{n-1}}$, since a function vanishing on $\mathbb{V}(x_0)$ is divisible by x_0 (by the Nullstellensatz). In other words, this is the ideal sheaf sequence. We can also think of it as coming from the exact sequence $0 \to S(-1) \to S \to S/(x_0) \to 0$. We can tensor with $\mathcal{O}_{\mathbb{P}^n}(d)$ and still get an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(d-1) \xrightarrow{\times x_0} \mathcal{O}_{\mathbb{P}^n}(d) \to i_*\mathcal{O}_{\mathbb{P}^{n-1}}(d) \to 0$$

Summing over all d, we get

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_{\mathbb{P}^{n-1}} \to 0$$

Now, by the long exact sequence and induction, we can easily show that $H^r(\mathbb{P}^n, \mathcal{F}) = 0, 0 < r < n$.

3.9 Differentials

The sheaf of differentials is the algebrogeometric analogue of the cotangent bundle.

3.9.1 Affine case

In the affine case, given a B-algebra A given by a map $B \to A$ and hence a morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$, we define the Kähler differentials to be the free A-module generated by symbols da satisfying the Leibniz rule and vanishing on B:

$$\Omega_{A/B} := \frac{\{da | a \in A\}}{db = 0, d(a+a') = da + da', d(aa') = a.da' + da.a}$$

Here, d is thought of as a B-linear derivation $d : A \to \Omega_{A/B}$. In fact, the differentials are the universal such derivation, in the sense that any other such object factors through an A-module map.

For example, if $A = B[x_1,...,x_n]$ then this is just $\oplus A dx_i$, which should be thought of as the differential forms on affine space. More generally, if $A = B[x_1,...,x_n]/(f_1,...,f_r)$ then $\Omega_{A/B} = \sum A dx_i/(df_j =$

0) and this is precisely the cokernel of the Jacobian matrix

$$J : A^r \to A^n$$

3.9.2 Global case

In order to globalize, we can look affine-locally and glue all the pieces together.

However, another approach is also possible. For this, we need the notion of a conormal sheaf.

Definition 3.74 (Conormal sheaf): *Let $Y \subset X$ be a subscheme cut out by the ideal sheaf $\mathcal{I}_Y$. Then we define the conormal sheaf by*

$$\mathcal{N}^*_{Y/X} := \mathcal{I}_Y / \mathcal{I}_Y^2$$

This should echo the fact that the two are isomorphic in the complex setting.

Now, given a morphism $\pi : X \to Y$ of schemes, we consider the diagonal map $\Delta : X \to X \times_Y X$ which was used to define separated morphisms. We then define

$$\Omega_{X/Y} := \mathcal{N}^*_{X/X \times_Y X}$$

We need to equip this with a derivation. Consider the diagram

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\
& X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\
& X \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

Then we define for a section f of $\mathcal{O}_X$ $df = p_1^* f - p_2^* f$. Notice that this is in the ideal sheaf $\mathcal{I}_X$ of sections vanishing on X since $\Delta^*(p_1^* f - p_2^* f) = f - f = 0$. Hence, we can consider it modulo the square of the ideal sheaf and hence obtain a map

$$\mathcal{O}_X \xrightarrow{d} \Omega_{X/Y}$$

Note that this is not a morphism of quasicoherent sheaves on X.

Example (Affine case again): Let's see what happens in the affine case. We have a map $\operatorname{Spec} A \to \operatorname{Spec} B$ of schemes, and hence on the level of rings the diagram looks like

$$\begin{tikzcd}
A & & \\
& A \otimes_B A \arrow[ul, "\alpha"'] & A \arrow[l] \arrow[ull, bend right] \\
& A \arrow[u] \arrow[uul, bend left] & B \arrow[l] \arrow[u]
\end{tikzcd}$$

The map α is given by multiplication. We see that the ideal sheaf is given by

$$I = \Gamma(\mathcal{I}_\Delta) = \{s \in \Gamma(\mathcal{O}_{\operatorname{Spec} A \otimes_B A}) | \Delta^* s = 0\} = \ker \alpha$$

This is generated by elements of the type $1 \otimes a - a \otimes 1$. We now see that the derivation takes the form $d : A \to I/I^2, a \mapsto 1 \otimes a - a \otimes 1$. One needs to check that this is well-defined, i.e. $d(aa') - ad(a') - a'(da) = (1 \otimes a - a \otimes 1)(1 \otimes a' - a' \otimes 1) \in I^2$. One then has to verify that $\Omega_{A/B} = I/I^2$. The isomorphism is given by $da \mapsto 1 \otimes a - a \otimes 1$ and inversely $x \otimes y \mapsto xdy$.

Thus we see that the global construction reduces to the situation from the previous section and naturally glues the affine local situation together.

3.9.3 Smoothness

We briefly mention a definition/criterion for smoothness of a scheme.

Definition 3.75 (Smoothness using differentials): *A k-scheme X is smooth if it is of finite type, pure dimension n and the sheaf of differentials $\Omega_{X/k}$ is locally free of rank n. This corresponds to the nonsingularity of the Jacobians that are used to define regularity in differential geometry.*

3.9.4 The Euler sequence

Theorem 3.76 (Euler sequence): *We have the following exact sequence:*

$$0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0$$

Proof. This has something to do with differentials on $A^{n+1} \setminus 0$ and Euler vector fields, but am not sure how it works exactly.

We begin with the local situation over the affine cover U_i. We put $\varphi(s_0, \ldots, s_n) = x_0 s_0 + \ldots + x_n s_n$ which sends a homogenous degree -1 tuple to a degree 0 tuple, so it makes sense as a map $\mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O}$.

On the other hand, over this open affine cover, the sheaf of differentials is generated by

$$f_0 d(\frac{x_0}{x_i}) + \ldots + f_n d(\frac{f_n}{x_i}) = \sum_{k \neq i} f_k d(\frac{x_k}{x_i})$$

We seek a way to make a differential form of this type into a tuple of sections of $\mathcal{O}(-1)$. However, if we actually think of the x_i as coordinates (this is where that comment about pulled back forms comes in) then

$$d(\frac{x_j}{x_i}) = \frac{1}{x_i} dx_k - \frac{x_j}{x_i^2} dx_i \implies x_j \frac{1}{x_i} + x_i \frac{-x_j}{x_i^2} = 0$$

This motivates the definition

$$\Omega_{\mathbb{P}^n_A/A}(U_i) \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus n+1}(U_i)$$
$$\sum_{j \neq i} f_k d(\frac{x_j}{x_i}) \mapsto ((\frac{f_0}{x_i}, \ldots, -\sum_{j \neq i} \frac{x_j}{x_i^2}) f_j, \ldots, \frac{f_n}{x_i})$$

We need to check that these glue, which is routine using the Leibniz rule, and that the image is precisely the kernel of φ. Given $(g_0, \ldots, g_n)$ with $\sum x_j g_j = 0$ we put $f_j = x_i g_j$ when $j \neq i$. Then $\sum f_j d(\frac{x_j}{x_i}) \mapsto (g_0, \ldots, g_n)$, so the map surjects onto the kernel. Furthermore, it is clearly injective, and this gives us the desired isomorphism.

$\square$

3.10 The Hodge numbers of projective space

Following Arupura - use cohomology of invertible sheaves on projective space, Euler sequence etc. to show that $h^{p,p} = 1$, otherwise 0.

3.11 Blowups

4 Algebraic Topology

4.1 Homology

4.1.1 Singular homology and the LES of a pair

Define n simplices as $\Delta^n = \{(t_0, \ldots, t_n) | \sum t_i = 1, t_i \geq 0\}$. For any $I \subseteq \{0, 1, \ldots, n\}$ we can associate the I-th face of Δ^n and furthermore a homeomorphism $F_I : \Delta^{|I|-1} \to f_I \subseteq \Delta^n$. This allows us to define a canonical chain complex $S_\bullet(\Delta^n)$ with trivial homology except in degree 0, such that S_k consists of the free abelian group on the k-faces. The boundary of a face is given by $df_I = \sum(-1)^j f_{I/\{j\}}$, the alternating sum of the faces of f_I. Moreover, this chain complex can be augmented by adding a face corresponding to the empty set and an augmentation map $\epsilon : \sum a_i f_i \mapsto \sum a_i$, giving the reduced homology groups which are all 0.

This canonical chain complex can be used to define the singular homology groups of any topological space X as follows:

Definition 4.1 (Singular chain complex): *Define $C_k(X)$ to be the free abelian group on the set of continous maps $\sigma : \Delta^k \to X$. Given any $k-1$ dimensional face of Δ^k given by some map $F : \Delta^{k-1} \to \Delta^k$, we can compose with σ to get a "face" of σ. Then the differential of σ can be defined once again as the alternating sum*

$$d\sigma = \sum (-1)^j \sigma \circ F_{\hat{j}}$$

This is chosen so that σ actually induces a chain map $\varphi_\sigma : S(\Delta^k) \to C(X)$. The homology of this chain complex is denoted $H_\bullet(X)$.

If we have a subspace $A \subset X$ the short exact sequence of chain complexes

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X)/C_\bullet(A) \to 0$$

induces the LES of a pair (X, A). The snake lemma tells us that the boundary map $H_\bullet(X, A) \to H_{\bullet-1}(A)$ takes a relative chain σ with boundary in A to the class $\partial\sigma$ which is a cycle.

4.1.2 Homotopy invariance of homology

We first describe a universal chain homotopy.

Firstly, note that for any $\sigma : \Delta^k \to X$, we have a map $\varphi_\sigma : S_\bullet(\Delta^k) \to C_\bullet(X)$ given by $\varphi_\sigma(f_I) = \sigma \circ F_I$. Now take two maps $\iota_0, \iota_1 : \Delta^n \to \Delta^n \times I$ which represent the bottom and top embeddings, i.e. $x \mapsto (x, 0)$ or $(x, 1)$. These give maps $\varphi_{\iota_0}, \varphi_{\iota_1} : S_\bullet(\Delta^n) \to C_\bullet(\Delta^n \times I)$. We want to show these are homotopic.

Some more definitions are due: for vectors $v_0, \dots, v_k$ in a convex set X, we define $[v_o, \dots, v_k] : \Delta^k \to X$ by $(t_i) \mapsto \sum t_i v_i$. This is the k-simplex spanned by these vectors, thought of as a map. Now $[v_0, \dots, v_k] \circ F_{\hat{j}} = [v_0, \dots, \hat{v_j}, \dots, v_k]$ and $d[v_0, \dots, v_k] = \sum (-1)^j [v_0, \dots, \hat{v_j}, \dots, v_k]$.

Now let's write i for the vertices of the bottom Δ^n in $\Delta^n \times I$ and i' for the vertices on the top. The idea is to triangulate $\Delta^n \times I$ and exhibit a chain homotopy operator.

The universal chain homotopy operator is a tool that interpolates between faces on the bottom to faces on the top. It is defined as follows: $U_n : S_\bullet(\Delta^n) \to C_{\bullet+1}(\Delta^n \times I)$ given by $U_n(f_I) = \sum (-1)^{j'} [i_0, ..., i_{j'}, i'_{j'}, ..., i'_k]$, where basically f_I is defined by $[i_0, \dots, i_k]$. This interpolates using simplices of 1 higher dimension than f_I, as we're adding a point and climbing up (see picture).

Proposition 4.2 (Proposition): $dU_n + U_n d = \varphi_{\iota_1} - \varphi_{\iota_0}$.

This comes down to showing that $(dU_n + U_n d)(f_I) = [i'_0, \dots, i'_k] - [i_0, \dots, i_k] = \varphi_{\iota_1}(f_I) - \varphi_{\iota_0}(f_I)$, and is done by a combinatorial argument (by looking at where we are splitting and what we are deleting).

It is good to note that this procedure is natural, i.e. doesn't depend unnaturaly on n. Formally, one can say that there is a commuting diagram of this type:

$$\begin{array}{ccc} S_\bullet(\Delta^k) & \xrightarrow{\varphi_I} & S_\bullet(\Delta^n) \\ {\scriptstyle U_k}\downarrow & & \downarrow{\scriptstyle U_n} \\ C_\bullet(\Delta^k \times I) & \xrightarrow[\overline{F_I}_*]{} & C_\bullet(\Delta^n \times I) \end{array}$$

Where $\varphi(f_J) = f_{i_{j_0} \dots i_{j_l}}$ with $J = \{j_0, \dots, j_l\}, I = \{i_0, \dots, i_k\}$ and $\overline{F_I}(x,t) = (F_I(x), t)$ so that $\overline{F_I}_*[j_0, \dots, j_l] = [i_{j_0} \dots i_{j_l}]$.

We can now use the universal chain homotopy to show that for any spaces X, Y and a homotopy $H : X \times I \to Y$ then the maps $f_0 \sim f_1$ induce the same maps on homology. One does this by defining for each $\sigma : \Delta^n \to X$ a corresponding $H_\sigma : \Delta^n \times I \to Y$ sending $(x,t) \mapsto H(\sigma(x), t)$. We have the following compositions, where H_σ is the composition of the second two maps:

$$\Delta^k \times I \xrightarrow{\overline{F_I}} \Delta^n \times I \xrightarrow{\sigma \times 1} X \times I \xrightarrow{H} Y$$

But note that $(\sigma \times 1) \circ \overline{F_I} = (\sigma \circ F_I) \times 1$, i.e. this is the same as

$$\Delta^k \times I \xrightarrow{(\sigma \circ F_I) \times 1} X \times I \xrightarrow{H} Y$$

With this notation, we can transport the universal chain homotopy:

$$S_n(\Delta^n) \xrightarrow{U_n} C_{n+1}(\Delta^n \times I) \xrightarrow{\sigma \times 1} C_{n+1}(X \times I) \xrightarrow{H} C_{n+1} Y$$

Now we can define a map $h : C_n(X) \to C_{n+1}(Y)$ by $\sigma \mapsto H_\sigma(U_n(f^n_{top}))$. It is now straightforward to verify that $dh + hd = f_{1*} - f_{0*}$:

$$hd(\sigma) = \sum (-1)^j H_{\sigma \circ F_{\hat{j}}} U_{n-1} f^{n-1} = \sum (-1)^j H_\sigma \overline{F_{\hat{j}}} U_{n-1} f^{n-1} = H_\sigma U_n d f^n$$

$$dh(\sigma) = dH_\sigma U_n f^n = H_\sigma d U_n f^n$$

So

$$(dh + hd)(\sigma) = H_\sigma(dU_n + U_n d)(f^n) = H_\sigma(\varphi_{\iota_1} - \varphi_{\iota_2})(f^n) = H_\sigma \circ \iota_1 - H_\sigma \circ \iota_2 = f_{1*}(\sigma) - f_{0*}(\sigma)$$

Hence homotopy equivalent spaces have the same homology groups etc.

4.1.3 Excision and subdivision

If we consider an open cover $\mathcal{U}$ and the chains $C_\bullet^{\mathcal{U}}(X)$ which are sums of chains landing in one of the open sets in the cover, then repeated barycentric subdivision can be used to show the following:

Lemma 4.3 (Barycentric subdivision): *The inclusion $\iota : C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$ is a homotopy equivalence and induces an isomorphism on homology.*

This allows us to prove many things, beggining with excision, but also later on Mayer-Vietoris.

Firstly, one can show that if $\mathcal{U} = \{U\}$ is an open cover of X and $\mathcal{U}^A$ is its restriction to A, then by the five lemma and the LES of a pair, we see that if we put $C^{\mathcal{U}} = C^{\mathcal{U}}(X)/C^{\mathcal{U}}(A)$, then $H_\bullet^{\mathcal{U}}(X,A) \simeq H_\bullet(X,A)$.

Proposition 4.4 (Excision): *$B \subset A \subset X$ with $\overline{B} \subset A^\circ$. Then the inclusion $j : (X - B, A - B) \to (X,A)$ induces an isomorphism on homology.*

Proof. Note that $\mathcal{U} = \{A^\circ, X/\overline{B}\}$ is an open cover of X. Now,

$C_\bullet^{\mathcal{U}}(X) = \langle \sigma \text{ subordinate to } \mathcal{U}\rangle = \langle \sigma | \mathrm{im}\sigma \cap B = \emptyset\rangle \oplus \langle \sigma | \mathrm{im}\sigma \cap B \neq \emptyset\rangle = C_\bullet^{\mathcal{U}}(X - B) \oplus \langle \sigma | \mathrm{im}\sigma \subset A, \mathrm{im}\sigma \cap B \neq \emptyset\rangle$.

Similarly,

$C_\bullet^{\mathcal{U}}(A) = \langle \sigma \text{ subordinate to } \mathcal{U}^A, \mathrm{im}\sigma \subset A\rangle = \langle \sigma | \mathrm{im}\sigma \cap B = \emptyset\rangle \oplus \langle \sigma | \mathrm{im}\sigma \cap B \neq \emptyset\rangle = C_\bullet^{\mathcal{U}}(A - B) \oplus \langle \sigma | \mathrm{im}\sigma \subset A, \mathrm{im}\sigma \cap B \neq \emptyset\rangle$.

So, $C_\bullet^{\mathcal{U}}(X,A) \simeq C_\bullet^{\mathcal{U}}(X - B, A - B)$. Hence, we get the following commutative diagram:

$$\begin{tikzcd}
H_\bullet^{\mathcal{U}}(X - B, A - B) \arrow[r, "j_*"] \arrow[d] & H_\bullet^{\mathcal{U}}(X,A) \arrow[d] \\
H_\bullet(X - B, A - B) \arrow[r, "j_*"'] & H_\bullet(X,A)
\end{tikzcd}$$

The vertical arrows are isos by barycentric subdivision, and the upper one is an iso by what we just did, so the lower one is iso as well. □

We can now use exision to prove the following:

Proposition 4.5 (Collapsing a pair): *If A is a deformation retract of a neighbourhood U in X (i.e. they form a good pair), then $H_\bullet(X,A) \simeq H_\bullet(X/A,A,A) \simeq \tilde{H}_\bullet(X/A)$.*

Proof. Now let's get back to collapsing a pair. Note that in the case of a good pair, we have $H_\bullet(X,A) \simeq H_\bullet(X,U)$ by the LES of a triple and the LES of a pair (U,A). Now we have the following:

$$\begin{tikzcd}
H_\bullet(X-A,U-A) \arrow[r, "j_*"] \arrow[d] & H_\bullet(X,U) \arrow[d] & H_\bullet(X,A) \arrow[l, "i_*"'] \arrow[d] \\
H_\bullet(X/A-A/A,U/A-A/A) \arrow[r, "j_*"'] & H_\bullet(X/A,U/A) & H_\bullet(X/A,A/A) \arrow[l, "i_*"']
\end{tikzcd}$$

The j arrows are isos by excision, and the i ones are isos by the what we just said. However, the quotient map $(X-A,U-A) \simeq (X/A-A/A,U/A-A/A)$ is a homeomorphism, hence the vertical left arrow is an iso, and so are all the other ones. $\square$

Definition 4.6 (Local homology): *By excision, $H_\bullet(M,M-m) \simeq H_\bullet(U,U-m)$ where U is some local, Euclidean neighbourhood of m. But then this is the same as $H_\bullet(D^n,D^n-0) = H_\bullet(D^n,\partial D^n) = H_\bullet(S^n)$ by collapsing a pair, or equivalently by LES of a pair. Hence, it is $\mathbb{Z}$ in dimension n. This can be shown that two manifolds of unequal dimension cannot be homeomorphic.*

4.1.4 Mayer-Vietoris

As another application of subdivision, we have the Mayer-Vietoris sequence. Given A,B covering X, we have that $C_\bullet(A+B) \sim C_\bullet(X)$ by subdivision. Hence, the SES of chain complexes

$$0 \to C_\bullet(A\cap B) \xrightarrow{(x,-x)} C_\bullet(A)\oplus C_\bullet(B) \xrightarrow{+} C_\bullet(A+B) \to 0$$

produces an LES.

4.1.5 Degrees

The homology $H_n(S^n)$ is, by Mayer-Vietoris for example, equal to Z and is generated by a cycle $\Delta_+ - \Delta_-$ where we think of S^n as the CW complex with two n-cells $D_\pm$ attached along an equator. Hence, any map $f : S^n \to S^n$ induces multiplication by an integer $\deg(f)$ on homology. This has the following properties:

- $\deg(f) = 0$ if f is not surjective.
- The degree is homotopy invariant and multiplicative
- The degree of reflection along S^{n-1} transposes $D_\pm$ and so has degree -1

- The degree of the antipodal map is $(-1)^{n+1}$ as it is obtained from $n+1$ reflections.
- If f has no fixed points, then it can be homotoped to the antipodal map via $f_t(x) = \frac{(1-t)f(x)-tx}{|(1-t)f(x)-tx|}$

To compute degree, we can use the tool of local degrees.

Definition 4.7 (Local degree): *By LES of pair $(S^n, S^n - p)$, we see that $H_n(S^n) \simeq H_n(S^n, S^n - p)$, $[S^n] \mapsto [S^n, S^n - p]$. Similarly, for any open $U \subset S^n$, we have, by excision, that $H_n(U, U-p) \simeq H_n(S^n, S^n - p)$ and we denote its fundamental class as $[U, U-p]$. Note that this is compatible with inclusions of opens $U' \subset U$.*

Suppose that $f^{-1}(p)$ consists of a finite number of points $\{q_1, \dots, q_n\}$. Then we want to show that the degree of f is the sum of the local degrees at each q_i - note that this is independent of the point p chosen, only on the fact that its preimage is finite!

Firstly, find separating neighbourhoods U_i of q_i and put $f_*[U_i, U_i - q_i] = (\deg_{q_i} f)[S^n, S^n - p]$, the local degree of f at q_i. Again, note that this is not really dependent on the choice of U_i as we can always restrict further and get the same result.

Now put $V = \coprod U_i$. By excision, all the groups in the diagram are isomorphic:

$$\begin{tikzcd}
H_n(V, V - f^{-1}(p)) \arrow[r, "\sim"] \arrow[d] & H_n(S^n, S^n - f^{-1}(p)) \arrow[d] \\
H_n(\coprod U_i, \coprod U_i - q_i) \arrow[r, "\sim"'] & \bigoplus H_n(U_i, U_i - q_i)
\end{tikzcd}$$

We want to show that the map $\beta : H_n(S^n) \to H_n(S^n, S^n - f^{-1}(p))$ is given by $[S^n] \mapsto \sum [U_i, U_i - q_i]$. However, we have the following diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[d] & \\
H_n(S^n, S^n - f^{-1}(p)) \arrow[r, "i_*"] \arrow[d, "\sim"'] & H_n(S^n, S^n - q_j) \arrow[d, "\sim"'] \\
H_n(V, V - f^{-1}(p)) \arrow[r] \arrow[d] & H_n(V, V - q_j) \arrow[d] \\
\oplus H_n(U_i, U_i - q_i) \arrow[r, "\pi_j"] & H_n(U_j, U_j - q_j)
\end{tikzcd}$$

If we follow the diagram around the left down and then right, we get $[S^n] \mapsto \pi_j \beta [S^n]$. On the other hand, if we follow the right, we get that this is simply $[U_j, U_j - q_j]$. But π_j is the projection, so we get the desired result.

Corollary 4.8 (Local degree formula): $\deg f = \sum \deg_{q_i} f$.

Proof. We inspect the following diagram:

$$\begin{tikzcd}
 & H_n(U_i, U_i - q_i) \arrow[r, "f_*"] \arrow[dl, "\simeq"'] \arrow[d] & H_n(V, V-p) \arrow[d, "\simeq"] \\
H_n(S^n, S^n - q_i) & H_n(S^n, S^n - f^{-1}(p)) \arrow[l] \arrow[r, "f_*"] & H_n(S^n, S^n - p) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\simeq"] \arrow[u] & H_n(S^n) \arrow[u, "\simeq"']
\end{tikzcd}$$

The top isos are from excision, the bottom ones from LES of pairs and all the unnamed maps come from inclusions. By excision, $H_n * S^n, S^n - f^{-1}(p)) \simeq \oplus H_n(U_i, U_i - q_i)$. The map $j : H_n(S^n) \to H_n(S^n, S^n - f^{-1}(p))$ takes a generator and projects it to a generator on the summands. Moreover, f_* acts by local degree on each of these separate generators, and hence we get the result. $\square$

4.1.6 Cellular homology

Cell complexes are built from the basic cells $(D^k, \partial D^k)$ by gluing along maps $f : \partial D^k \to X_{k-1}$. Their k-skeleta provide a filtration $X_0 \subset X_1 \subset X_2 \ldots$, which can be used to calculate homology groups. Note, firstly, that (X_k, X_{k-1}) is a good pair and $X_k/X_{k-1} \simeq \vee S^k$ is a wedge sum of spheres, precisely because $D^k/\partial D^k \simeq S^k$.

Example (Cell structure on real and complex projective spaces): Note that S^n can be built from 2 cells in each dimension (2 points, then 2 lines, then two disks for upper and lower hemisphere etc.). But $\mathbb{RP}^n \simeq S^n/\mathbb{Z}/2$, so we can get a cell structure on $\mathbb{RP}^n$ by identifying the cells in pairs, getting one cell in each dimension. Another way to think about this is $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_p D^n$, where $p : S^{n-1} \to \mathbb{RP}^{n-1}$ is the projection map.
For complex projective space $\mathbb{CP}^n = \mathbb{C}^{n+1} - 0/\mathbb{C}^\times$, note that $\mathbb{C}^\times \simeq \mathbb{R}_{\geq 0} \times S^1$, $\mathbb{C}^{n+1} - 0/\mathbb{R}_{\geq 0} \simeq S^{2n+1}$, so $\mathbb{CP}^n \simeq S^{2n+1}/S^1$. Let $q : S^{2n+1} \to \mathbb{CP}^n$ be the Hopf quotient map. Then, we claim that $\mathbb{CP}^n = \mathbb{CP}^{n-1} \cup_q D^{2n}$. To see this, note that $\mathbb{CP}^{n-1}$ embeds in $\mathbb{CP}^n$ using $[z] \mapsto [z : 0]$. Furthermore, we have a map $D^{2n} \to \mathbb{CP}^n, z \mapsto [z : \sqrt{1 - \|z\|^2}]$ such that the boundary ∂D^{2n} maps into exactly the bit we are identifying with $\mathbb{CP}^{n-1}$. In fact, we get the following:

$$S^{2n-1} = \partial D^{2n} \subset D^2 n \to \mathbb{CP}^n z \mapsto [z : 0]$$

In particular, if we identify the image of S^{2n-1} with $\mathbb{CP}^{n-1}$, we get precisely $z \mapsto [z]$, which is the Hopf map. Hence our two maps glue to give $\mathbb{CP}^{n-1} \cup_q D^2 n \to \mathbb{CP}^n$. This is an isomorphism, as if $[z_0 : \ldots : z_n] \in \mathbb{CP}^n$, then if $z_n \neq 0$, it corresponds to a normalized $(z_0, \ldots, z_{n-1}) \in D^2 n$, and if $z_n = 0$, we just get $[z_0 : \ldots : z_{n-1}] \in \mathbb{CP}^{n-1}$.
Now we know that $\mathbb{CP}^n/\mathbb{CP}^{n-1} \simeq S^{2n}$ and using the LES in homology, we can calculate that $H_\bullet(\mathbb{CP}^n) = \mathbb{Z}$ if $\bullet$ is even and 0 otherwise.

> **Definition 4.9 (The cellular complex):** *Given a fcc X, we can combine LES's of pairs to get a composition*
>
> $$H_k(X_k, X_{k-1}) \xrightarrow{\partial} H_{k-1}(X_{k-1}) \xrightarrow{j} H_{k-1}(X_{k-1}, X_{k-2})$$
>
> *The first comes from the long exact sequence of the pair (X_k, X_{k-1}) and the second from the long exact sequence of the other pair. In other words, we are taking a relative cycle, taking its boundary so it lies in X_{k-1} and then quotienting to get a relative cycle again. The resulting map is called d_k, the boundary map in the cellular chain complex. Note that it is indeed a chain complex, as a composition of two differentials is really 4 maps in 2 long exact sequences and the middle two compose to 0.*

A few remarks: X is a fcc with 1 0-cell and all other cells of dimension $\in [m, M]$, then the only nontrivial reduced homology occurs in those dimensions. $H_k(X, X_k) = 0$ and hence $H_k(X_{k+1}) = H_k(X)$

We can use this to show that cellular homology is isomorphic to our usual homology, using the following big double complex:

$$\begin{tikzcd}
 & & & & 0 = H_k(X_{k+1}, X_k) & \\
 & 0 = H_k(X_{k-1}) \arrow[dr] & & H_k(X_{k+1}) \arrow[ur] & & \\
 & & H_k(X_k) \arrow[ur, two heads, "\iota"] \arrow[dr, tail, "\pi"] & & & \\
\dots \arrow[r] & H_{k+1}(X_{k+1}, X_k) \arrow[ur, "\partial"] \arrow[rr, "d"] & & H_k(X_k, X_{k-1}) \arrow[dr, "\partial"] \arrow[rr, "d"] & & H_{k-1}(X_{k-1}, X_{k-2}) \\
 & & & & H_{k-1}(X_{k-1}) \arrow[ur, tail, "\pi"] & \\
 & & & 0 = H_{k-1}(X_{k-2}) \arrow[ur] & &
\end{tikzcd}$$

We can identify $\ker d_k = \ker \partial = \operatorname{im} \pi \simeq H_k(X_k)$. On the other hand, $\operatorname{im} d_{k+1} = \operatorname{im} \partial = \ker \iota$. But ι is surjective and finally

$$H_k^{cell}(X) = \ker d_k / \operatorname{im} d_{k+1} = H_k(X_k)/\ker \iota = \operatorname{im} \iota = H_k(X_{k+1}) = H_k(X)$$

Now we turn to the question of calculating the homology of the cellular complex. As mentioned before, X_k/X_{k-1} is just a wedge sum of circles, so each $H_k(X_k.X_{k-1})$ is free abelian on the set of k-cells. If e_α denote the k-cells and e_β the $k-1$-cells, then the attaching map of e_α gives us the following information:

$$\partial D^k \xrightarrow{f_\alpha} X_{k-1} \xrightarrow{\pi_{k-1}} X_{k-1}/X_{k-2} \simeq \vee S^{k-1} \xrightarrow{p_\beta} S^{k-1}$$

We want to show that the matrix for d_k with respect to the bases e_α and e_β is given by the degrees of the maps $p_\beta \pi_{k-1} f_\alpha = f_{\alpha\beta}$. Note that p_β quotients out all of the other irrelevant spheres.

Denote $\iota_\alpha : (D^k, S^{k-1}) \to (X_k, X_{k-1})$ the result of attaching e_α to X_{k-1}. By the LES of a pair and collapsing a pair, the class e_α that generates a piece of X_k/X_{k-1} we are interested in is the same as the image $\iota_{\alpha*}[D^k, S^{k-1}] \in H_k(X_k, X_{k-1})$. Also, ι_α is a map of pairs, so induces a map on their long exact sequences, meaning that it commutes with the boundary map as follows:

$$\begin{tikzcd} H_k(D^k, S^{k-1}) \arrow[r, "\partial_k"] \arrow[d, "\iota_*"'] & H_{k-1}(S^{k-1}) \arrow[d, "\iota_*"'] \\ H_k(X_k, X_{k-1}) \arrow[r, "\partial_k"] & H_{k-1}(X_{k-1}) \end{tikzcd}$$

Hence:

$$d_k(e_\alpha) = \pi_{k-1*}\partial_k\iota_{\alpha*}[D^k, S^{k-1}] = \pi_{k-1*}\iota_{\alpha*}\partial_k[D^k, S^{k-1}] = \pi_{k-1*}\iota_{\alpha*}[S^{k-1}] = \pi_{k-1*}f_{\alpha*}[S^{k-1}]$$

Now, this lives in $H_{k-1}(X_{k-1}, X_{k-2})$, which is a direct sum of $\mathbb{Z}$'s, so to get the integer corresponding to β, we must project using p_β, getting $p_{\beta*}\pi_{k-1*}f_{\alpha*}[S^{k-1}] = f_{\alpha\beta*}[S^{k-1}] = \deg f_{\alpha\beta}[S^{k-1}]$, and this implies that $d_k(e_\alpha) = \sum \deg f_{\alpha\beta} e_\beta$.

Example (Cellular homology of projective space): As mentioned, $\mathbb{RP}^n$ has 1 cell in each dimension, so we get a complex $\mathbb{Z} \to \mathbb{Z} \to \dots$ We need to calculate $d_k(e_k)$ in terms of e_{k-1}, which requires calculating the degree of the composite map

$$S^{k-1} \xrightarrow{p} \mathbb{RP}^{k-1} \to \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \simeq S^{k-1}.$$

But picking $x \in \mathbb{RP}^{k-1} - \mathbb{RP}^{k-2}$, it has two distinct preimages in S^{k-1}, namely a point q and its antipode Aq. However, $p = p \circ A$, hence $\deg_{Aq}(p) = \deg_q(p)\deg(A) = \deg_q(p)(-1)^k$. However, p is a local homeomorphism, so $\deg_q(p) = 1$. All in all, the total degree is the sum of the local degrees, which is 0 or 2, depending on whether k is odd or even.

Remark (CW pairs): Note that a map of CW complexes f induces maps on cellular homology and it coincides with the one given in singular homology. Moreover, if $A \subset X$ is a CW pair, then we have a short exact sequence of complexes

$$0 \to H_k(A_k, A_{k-1}) \to H_k(X_k, X_{k-1}) \to H_k(X_k, X_{k-1} \cup A_k) \to 0$$

This follows by just observing that if $e_1, \dots, e_n$ is the set of k-cells of A and $e_1, \dots, e_m$ is the set of k-cells of X then $X_k/X_{k-1} \cup A_k = e_{n+1} \vee \dots \vee e_m$. Hence, we have a relative cellular homology complex which just recovers the relative singular homology. We can check the inclusions form chain maps by looking at the diagram

$$\begin{tikzcd} H_k(X_k, X_{k-1}) \arrow[r] \arrow[d, "\delta"'] & H_k(X_k, X_{k-1} \cup A_k) \arrow[d, "\delta"] \\ H_{k-1}(X_{k-1}) \arrow[r] \arrow[d, "\pi"'] & H_{k-1}(X_{k-1} \cup A_k) \arrow[d, "\pi"] \\ H_{k-1}(X_{k-1}, X_{k-2}) \arrow[r] & H_{k-1}(X_{k-1}, X_{k-} \cup A_{k-1}) \end{tikzcd}$$

4.1.7 Examples and calculations of homology

Example (LES of cone): Suppose $f : X \to Y$ is a map of spaces with cone C_f. We will show that upon applying homology, we get an exact triangle. One way to do this is to consider M_f the mapping cylinder, which is homotopy equivalent to Y and the LES of pairs (Y, M_f) gives the desired LES, since $M_f/Y \simeq X$. Another way to do this is directly use the LES of (Y, C_f) and the fact that $C_f/Y \simeq \Sigma X$. What we need to verify is that the boundary map $H_{\bullet+1}(C_f, Y) \to H_\bullet(Y)$ corresponds to $f_* : H_\bullet(X) \to H_\bullet(Y)$ under the suspension degree shift isomorphism. In other words, we would like the following to commute:

$$\begin{tikzcd}
H_{\bullet+1}(C_f, Y) \arrow[r, "q_*"] \arrow[dr, "\delta"'] & H_{\bullet+1}(C_f/Y, Y/Y) \arrow[r, "\simeq"] & \tilde{H}_{\bullet+1}(\Sigma X) \arrow[r, "\simeq"] & \tilde{H}_{\bullet+1}(CX/X) \arrow[r, "\simeq"] & H_{\bullet+1}(CX, X) \arrow[dl, "\delta"] \\
& H_\bullet(Y) & & H_\bullet(X) \arrow[ll, "f_*"] &
\end{tikzcd}$$

But note that we have a commuting diagram

$$\begin{tikzcd}
H_\bullet(C_f, Y) \arrow[r, "q_*"] & \tilde{H}_\bullet(C_f/Y) \arrow[d, "\simeq"] \\
H_\bullet(CX, X) \arrow[u, "(Cf_*, f_*)"] \arrow[r, "q_*"] & \tilde{H}_\bullet(CX/X)
\end{tikzcd}$$

Hence, Cf_* provides the inverse isomorphism and we are reduced to showing the following commutes:

$$\begin{tikzcd}
& H_{\bullet+1}(CX, X) \arrow[dl, "(Cf_*, f_*)"'] \arrow[dr, "\delta"] & \\
H_{\bullet+1}(C_f, Y) \arrow[dr, "\delta"'] & & H_\bullet(X) \arrow[dl, "f_*"] \\
& H_\bullet(Y) &
\end{tikzcd}$$

But this is true by definition of the boundary maps.

Example (Even dimensional projective space): Suppose $f : S^{2n} \to S^{2n}$ is a map. Then, if $f(x) \neq \pm x$ for all x, then we would be able to homotope f into both the identity and antipodal maps, which have different degrees, by normalizing the formula $tf(x) + (1-t) \pm x$. This is impossible, hence any such map has some x with $f(x) = \pm x$. This also shows that any map $\mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

Example (Cellular homology of some projective-like spaces): Let X be obtained from S^n by collapsing $x \sim -x$ for $x \in S^{n-1}$ on the equator. This can also be described as the CW complex with a 0-cell, one $n-1$ cell and two n-cells attached along the identity and antipodal map respectively. Clearly $H_0(X) \simeq \mathbb{Z}$ and the reduced cellular complex is then

$$0 \to \mathbb{Z}^2 \xrightarrow{d} \mathbb{Z} \to 0$$

Let the n-cells be e_+, e_- and the $n-1$-cell be e. Thus, $de_+ = e, de_- = \deg(A)e = (-1)^{n+1}e$. When n is even, the map is thus $(1,-1)$ and hence $H_n(X) \simeq \mathbb{Z}(e_+ + e_-)$ and $H_{n-1}(X) = 0$. On the other hand, when n is odd, we have that $H_n(X) = 0$ and $H_{n-1}(X) = 0$. This has to do with the fact that only the even-dimensional projective space is $\mathbb{Z}$-orientable. If we use $\mathbb{Z}_2$ coefficients, we see that in both cases $H_n(X;\mathbb{Z}_2) = \mathbb{Z}_2$, which is consistent with the fact that all manifolds are $\mathbb{Z}_2$-orientable.

Example (Even maps): The degree of the quotient map $S^n \to \mathbb{RP}^{2n}$ is 2 when n is odd and 0 when n is even, which can be used to show that an even map has even degree, since it factors through $\mathbb{RP}^n$.

Example (Complements of embeddings in spheres): We have that $\tilde{H}_i(S^n - h(D^k)) = 0$ for all i and any embedding $h : D^k \to S^k$. Moreover, if $k < n$ then $\tilde{H}_i(S^n - h(S^k))$ is zero, except for $i = n-k-1$ where it is $\mathbb{Z}$.

To show the first bit, one does a Mayer-Vietoris inductive argument on the sets $A = S^n - h(I^{k-1} \times [0,1/2]), A = S^n - h(I^{k-1} \times [1/2,1])$ replacing D^k by I^k as it is more convenient. The details can be found in Hatcher.

For the second part, one again does Mayer-Vietoris for the decomposition of S^k into two hemispheres and hence has $A, B = S^n - h(D^k_\pm)$ with trivial reduced homology by the first part and hence Mayer-Vietoris gives $\tilde{H}_i(S^n - h(S^k)) \simeq \tilde{H}_{i+1}(S^n - h(S^{k-1}))$.

4.1.8 Universal coefficient theorem for homology

We have a short exact sequence of chain complexes as follows:

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

The differentials on $Z_\bullet, B_\bullet$ are trivial. The rows split, as B_{n-1} is free, so the sequence remains exact after tensoring with G, i.e. we get a SES of chain complexes $0 \to Z_\bullet \otimes G \to C_\bullet \otimes G \to B_\bullet[-1] \otimes G \to 0$. The LES of this SES of chain complexes is given by

$$\ldots \longrightarrow Z_n \otimes G \longrightarrow H_n(C_\bullet; G) \longrightarrow B_{n-1} \otimes G \xrightarrow{\delta} Z_{n-1} \otimes G \longrightarrow \ldots$$

since the differentials on $Z_\bullet \otimes G, B_\bullet \otimes G$ are 0. The boundary map δ is precisely given by the inclusion maps $\iota \otimes 1$. Hence, we can break up this LES into a SES:

$$0 \to \operatorname{coker}(\iota_n \otimes 1) \to H_n(C_\bullet; G) \to \ker(\iota_{n-1} \otimes 1) \to 0$$

Since tensor product is right exact, we have that $\operatorname{coker}(\iota_n \otimes 1) \simeq H_n(C_\bullet) \otimes G$, so we only need to understand the kernel. To do this, we consider a different short exact sequence, which interprets it as a Tor group.

In other words,

$$0 \to B_n \xrightarrow{\iota_n} Z_n \to H_n(C_\bullet) \to 0$$

is a free resolution of $H_n(C)$ and hence we get a long exact sequence after tensoring with G, since Tor measures the failure of exactness:

$$0 \to \operatorname{Tor}_1(H_n(C), G) \to B_n \otimes G \to Z_n \otimes G \to H_n(C_\bullet) \otimes G \to 0$$

The first zero comes from the fact that $\operatorname{Tor}_1(Z_n, G) = 0$, as Z_n is free, and similarly with B_n. Hence, $\ker(\iota_n \otimes 1) \simeq \operatorname{Tor}_1(H_n C_\bullet, G)$ and the SES above produces:

Theorem 4.10 (Universal coefficient theorem for homology): *We have a short exact sequence*

$$0 \to H_n(C_\bullet) \otimes G \to H_n(C_\bullet; G) \to \operatorname{Tor}_1(H_{n-1}(C_\bullet), G) \to 0$$

These split, but not naturally, however we can still describe the above as

$$H_n(C_\bullet; G) \simeq \operatorname{Tor}_0(H_n(C_\bullet), G) \oplus \operatorname{Tor}_1(H_{n-1}(C_\bullet), G)$$

Remark: Importantly, Tor is symmetric, commutes with direct sums, is zero for free modules. Moreover, $\operatorname{Tor}_1(\mathbb{Z}_n, A) \simeq A/nA$. For finitely generated A, B, we have that $\operatorname{Tor}_1(A, B)$ is the tensor product of the torsion subgroups of A and B.

Corollary 4.11 (Corollary): $\tilde{H}_n(X; \mathbb{Z}) = 0$ *for all n if and only if* $\tilde{H}_n(X; \mathbb{Q}) = \tilde{H}_n(X; \mathbb{Z}_p) = 0$ *for all prime p. Hence, by using the mappinc cone, a map $f : X \to Y$ induces an isomorphism on integral homology if and only if it induces isomorphisms on rational and mod p homology.*

Proof. The UCT gives the only if half. For the if part, suppose we have a group A with $A \otimes \mathbb{Q} = 0, \operatorname{Tor}_1(A, \mathbb{Z}_p) = 0$. Then, use the short exact sequences $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and tensor with A to get Tor LES's. $\square$

4.2 Cohomology

Formal dual of homology, need to include Universal Coefficient theorem.

Definition 4.12 (Pairing between homology and cohomology): $\mathrm{Hom}(C_\bullet(X), G) \otimes C_\bullet(X) \to G$ *given by* $\alpha, c \mapsto \alpha(c)$*. This descends to cohomology:*

$$(\alpha + d^*\beta)(c + db) = \alpha(c) + \alpha(db) + (d^*\beta)(c) + (d^*\beta)(db) = \alpha(c) + (d^*\alpha)(b) + \beta(dc) + \beta(d^2 b) = \alpha(c)$$

4.2.1 Universal Coefficient theorem for cohomology

Similarly as in the UCT for homology, we will utilize two different short exact sequences and do some homological yoga.

Again, let's take the SES

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

Homming into G gives a SES and an associated LES in homology:

$$\dots \longleftarrow B_n^* \xleftarrow{\delta} Z_n^* \longleftarrow H^n(C_\bullet; G) \longleftarrow B_{n-1}^* \longleftarrow \dots$$

The boundary map is the dual of the inclusion map (by chasing the definitions) and hence we can break this up into a short exact sequence

$$0 \leftarrow \ker \iota_n^* \leftarrow H^n(C_\bullet; G) \leftarrow \mathrm{coker}\, \iota_{n-1}^* \leftarrow 0$$

However, $\ker \iota_n^* \simeq \mathrm{Hom}(H_n(C_\bullet), G)$ and hence we ony need to understand the cokernel. Take the free resolution

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C_\bullet) \to 0$$

After homming into G, the Ext LES gives us:

$$0 \to \mathrm{Hom}(H_{n-1}(C_\bullet), G) \to \mathrm{Hom}(Z_{n-1}, G) \to \mathrm{Hom}(B_{n-1}, G) \to \mathrm{Ext}^1(H_{n-1}(C_\bullet), G) \to 0$$

This describes the cokernel as an Ext group and hence:

Theorem 4.13 (Universal coefficient theorem for cohomology): *We have a short exact sequence*

$$0 \to \mathrm{Ext}^1(H_{n-1}(C_\bullet), G) \to H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet), G) \to 0$$

4.2.2 Cup product

Definition 4.14 (Cup product): *If the coefficient group is actually a ring, then the cup product on cochains is defined as*

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma \circ F_{0\ldots k})\beta(\sigma \circ F_{k\ldots k+l})$$

The map $F_{0\ldots k} : \Delta^k \to \Delta^{k+l} : (x_0, \ldots, x_k) \mapsto (x_0, \ldots, x_k, 0, \ldots, 0)$ *and similarly* $F_{k\ldots k+l} : \Delta^l \to \Delta^{k+l} : (x_0, \ldots, x_l) \mapsto (0, \ldots, 0, x_0, \ldots, x_l)$.

Note that this operation is associative and distributive, but not commutative! Furthermore, it obeys the Leibniz rule

$$d^*(\alpha \cup \beta) = d^*\alpha \cup \beta + (-1)^{|\alpha|}\alpha \cup d^*\beta$$

The proof of this goes by unravelling the definitions and doing some combinatorics with the indices. As a corollary to the Leibniz rule, we see that the cup product descends to cohomology, and it also commutes with pullback. the main property which separates the cup product on cochains and cohomology is that it is actually graded commutative on cohomology:

Proposition 4.15 (Associativity of cup product on cohomology): *On the level of cohomology,*

$$\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha$$

Proof. Proceed by constructing a map on chains $r : C_\bullet(X) \to C_\bullet(X)$ which swaps things around, i.e. from the linear map $\rho_n : \Delta^n \to \Delta^n, e_i \mapsto e_{n-i}$, get $r_j : C_j(X) \to C_j(X), \sigma \mapsto (-1)^{j(j+1)/2}\sigma \circ \rho_j$. By showing that this map is chain homotopic to the identity, we will be able to show the result, since $r[\alpha \cup \beta] = \pm[\beta] \cup [\alpha]$ on the one hand, since r swaps things around, but also by compatibility with the cup product, $r[\alpha \cup \beta] = r[\alpha] \cup r[\beta] = \pm[\alpha] \cup [\beta]$.

Hence, what we need to show is that r is a chain map and is chain homotopic to the identity. This can be done by induction on cell complexes, as in the lecture notes. Another way is to construct an explicit chain homotopy, as Hatcher does. The reversal map takes a n-simplex $[v_0, \ldots, v_n]$ and spits out the reversed n-simplex $[v_n, \ldots, v_0]$. In other words,

$$(r^*\varphi \cup r^*\psi)(\sigma) = \varphi(\epsilon_k \sigma|[v_k, \ldots, v_0])\psi(\epsilon_l \sigma|[v_l, \ldots, v_0])$$
$$r^*(\psi \cup \varphi)(\sigma) = (\psi \cup \varphi)(\sigma \circ r) = \epsilon_{k+l}\psi(\sigma|[v_l, \ldots, v_0])\varphi(\sigma|[v_k, \ldots, v_0])$$

Since r^* is chain homotopic to the identity, and the ring is commutative, we get the desired graded commutativity on cohomology. To take care of the chain map property, note that

$$dr(\sigma) = \epsilon_n \sum (-1)^i \sigma|[v_n, \ldots, \hat{v}_{n-i}, \ldots, v_0]$$
$$rd(\sigma) = r(\sum (-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n]) =$$
$$= \epsilon_{n-1} \sum (-1)^{n-i} \sigma|[v_n, \ldots, \hat{v}_{n-i}, \ldots, v_0]$$

Using the subdivision of $\Delta^n \times I$ using $[v_0, ..., v_n]$ on $\Delta^n \times \{0\}$ and the opposite orientation $[w_n, ..., w_0]$ on $\Delta^n \times \{1\}$, we can define a map $P : C_n(X) \to C_{n+1}(X)$ as

$$P(\sigma) = \sum (-1)^i \epsilon_{n-i}(\sigma\pi)[v_0, ..., v_i, w_n, ..., w_i],$$

where $\pi : \Delta^n \times I \to \Delta^n$ is the projection. Really, we have an element E_n in $C_{n+1}(\Delta^n \times I)$ and we are pushing it forward using π and $\sigma : \Delta^n \to X$, i.e. this is

$$(\sigma \circ \pi)_*(\sum (-1)^i \epsilon_{n-i}[v_0, ..., v_i, w_n, ..., w_i])$$

Now,

$$dP = (\sigma \circ \pi)_* dE_n = \sum_{j \le i} (-1)^i (-1)^j \epsilon_{n-i}[v_0, ..., \hat{v}_j, ..., v_i, w_n, ..., w_i]$$
$$+ \sum_{j \ge i} (-1)^i (-1)^{i+1+n-j} \epsilon_{n-i}[v_0, ..., v_i, w_n, ..., \hat{w}_j, ..., w_i]$$

When $j = i$, we get precisely $r(\sigma) - \sigma$, corresponding to $i = j = 0$ and $i = j = n$, with the other ones canceling in pairs and totalling to 0. Moreover, when $j \ne i$ we get $-Pd$, hence $Pd + dP = r - 1$. This is since

$$P(d\sigma) = (d\sigma \circ \pi)_* E_{n-1} = P(\sum (-1)^j [v_0, ..., \hat{v}_j, ..., v_n]) =$$
$$= \sum_{j<i} (-1)^{i-1} (-1)^j \epsilon_{n-i}[v_0, ..., \hat{v}_j, .., v_i, .w_n, ..., w_i]$$
$$+ \sum_{j>i} (-1)^i (-1)^j \epsilon_{n-i-1}[v_0, ..., v_i, w_n, ..., \hat{w}_j, ..., w_i]$$

The $(-1)^{i-1}$ comes up since we have $i-1$ elements $v_0, ..., \hat{v}_j, ..., v_i$ before the w_n shows up, and similarly for the ϵ_{n-i-1}. This cancels with the $j \ne i$ terms of dP, and we are done. □

Remark: using subdivision, one can show that relative cup product gives a map $H^\bullet(X, A) \times H^\bullet(X, B) \to H^\bullet(X, A \cup B)$. Note also that the cohomology ring of a disjoint union is the direct sum of the cohomology rings. Furthermore, $H^\bullet(X, p) = \bigoplus_{i>0} H^i(X)$. Moreover, $H^i(X \vee Y) = H^i(X) \oplus H^i(Y), i > 0$ and $\mathbb{Z}, i = 0$.

4.2.3 The exterior product and the Künneth formula

Given two spaces X, Y we have projection maps $p_X : X \times Y \to X, p_Y : X \times Y \to Y$. Hence, by pulling back, we get a bilinear map, i.e. a map

$$H^\bullet(X) \otimes H^\bullet(Y) \to H^\bullet(X \times Y)$$

given by $a \otimes b \mapsto a \times b := p_X^* a \cup p_Y^* b$. More generally, we have a map

$$H^\bullet(X, A) \otimes H^\bullet(Y) \xrightarrow{\mu} H^\bullet(X \times Y, A \times Y)$$

for a subspace $A \subset X$. If we equip the tensor product with a multiplication $(a \otimes b) \cup (c \otimes d) = (-1)^{|b||c|}(a \cup c \otimes b \cup d)$, then this becomes a ring homomorphism:

$$\mu((a \otimes b)(c \otimes d)) = (-1)^{|b||c|} \mu(ac \otimes bd) = (-1)^{|b||c|} p_X^*(a \cup c) \cup p_Y^*(b \cup d) =$$
$$= (-1)^{|b||c|} p_X^*(a) \cup p_X^*(c) \cup p_Y^*(b) \cup p_Y^*(d) = p_X^*(a) \cup p_Y^*(b) \cup p_X^*(c) \cup p_Y^*(d) =$$
$$= \mu(a \otimes b)\mu(c \otimes d)$$

We can ask whether this ring homomorphism is an isomorphism, which has an answer in the following theorem:

Theorem 4.16 (Kunneth formula): *μ is an isomorphism of rings if X and Y are cell complexes and $H^k(Y;R)$ is a finitely generated and free R–module for all k.*

Proof. Consider the functor $\overline{h}(X,A) = H^\bullet(X\times Y, A\times Y)$ such that, when $f:(X,A)\to(X',A')$ we have $\overline{f}: H^\bullet(X'\times Y, A'\times Y)\to H^\bullet(X\times Y, A\times Y)$ induced by $f\times 1_Y$. Moreover, consider $\underline{h}(X,A) = H^\bullet(X,A)\otimes H^\bullet(Y) = \bigoplus H^i(X,A)\otimes H^{n-i}(Y)$ and $\underline{f} = f^*\otimes 1_Y$. We want to show that these define cohomology theories and are isomorphic via the exterior product when the conditions of the theorem are met, which is done by comparing them on cell complexes and doing induction. They obviously satisfy homotopy invariance. For LES of a pair for $\overline{h}$, that is the LES of $(X\times Y, A\times Y)$, whereas for $\underline{h}$ it is the LES of (X,A) tensored with the free, hence flat, R-module $H^\bullet(Y)$. Naturality of the exterior product is on the example sheet. Now, the point is that a natural transformation between cohomology theories, which is an isomorphism on the pair $(point, \emptyset)$, then it is an isomorphism on all CW pairs (Hatcher, proposition 3.17). In other words, we can now use the five lemma, the LES property and induction, and that's it. □

Example: the cohomology ring of a genus g surface Σ_g has $H^\bullet(\Sigma_g)$ generated by a_i, b_i with the only nontrivial relation being $a_i\cup b_i = c$ such that c generates H^2.

4.3 The Thom isomorphism and Poincare duality

4.3.1 Thom classes

An R-Thom class is an element $u\in H^n(E,E^\#;R)$ which restricts to a generator on each fiber. For example, on a trivial bundle $E = B\times\mathbb{R}^n$ the Kunneth formula tells us that

$$H^\bullet(B\times\mathbb{R}^n, B\times(\mathbb{R}^n\setminus 0)) \simeq H^\bullet(B)\otimes H^\bullet(\mathbb{R}^n,\mathbb{R}^n\setminus 0)$$

and so taking the external product with the generator of the latter ring gives an isomorphism $H^{\bullet-n}(B)\simeq H^\bullet(E,E^\#;R)$. The Thom isomorphism states that this holds for all bundles which have a Thom class.

Proposition 4.17 (Properties of Thom classes): *If u is a Thom class for E, then f^*u is a Thom class for f^*E. Moreover, Thom classes glue: if u_1, u_2 are Thom classes of E_{B_1}, E_{B_2} agreeing on the intersection, then they glue to a Thom class on the union.*

Using this, one can use a Mayer-Vietoris argument to show the Thom isomorphism holds, and that every bundle has a $\mathbb{Z}/2$-Thom class.

Definition 4.18 (Euler class): *We define $e(E) = s_0^* j^* u \in H^n(B)$.*

Using the Gysin sequence, one can then show that $H^\bullet(\mathbb{RP}^n;\mathbb{Z}/2) \simeq \mathbb{Z}/2[x]/(x^{n+1})$ where $x = e(\tau)$, and similarly for $\mathbb{CP}^n$.

Example (Complex bundles are orientable): Given $E \to B$ with complex fibers, with transition functions in $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$, we want to show there is a Thom class. This amounts to showing that there is a canonical orientation on $\mathbb{C}^n$ which is preserved under complex linear maps. But the idea is that the determinant of A, thought of as a real linear map, is $|A_\mathbb{R}| = |A_\mathbb{C}|^2$ as in the situation in complex geometry, where the C-R equations imply that it is similar to a matrix with two blocks. In other words, the degree of the complex linear map $A : H^n((\mathbb{C}^n, \mathbb{C}^n \setminus 0) \to H^n((\mathbb{C}^n, \mathbb{C}^n \setminus 0)$ is given by the determinant of A, which is positive.

This comes down to the fact that any linear map induces its determinant - since there are two path-components of GL, one only need to check this for $\pm I$.

Definition 4.19 (Orientability): *A manifold is R-orientable if there is a class $[M] \in H_n(M)$ which restricts to a generator on the local homology groups $H_n(M, M - x)$. We have that M is orientable iff TM is orientable, by looking at tubular neighbourhoods of curves in M.*

4.3.2 Poincare duality

Definition 4.20 (Cap product): $H_{k+l}(M) \to H_l(M), x \mapsto x \cap a$ *is dual to the cup product, where $a \in H^k(M)$. In other words, it is an operation*

$$H_{k+l}(M) \otimes H^k(M) \to H_l(M)$$

On chains, this basically contracts a cochain ϕ by the chain σ:

$$\sigma \cap \phi = \phi(\sigma|[v_0, ..., v_k])\sigma|[v_k, ..., v_{k+l}]$$

This also satisfies the projection formula

$$f_*(\alpha) \cap \phi = f_*(\alpha \cap f^*\phi*)$$

We have the adjoint relation

$$\langle a \cup b, x\rangle = \langle b, x \cap a\rangle$$

Moreover, we can define an intersection pairing

$$H^k(M) \otimes H^{n-k}(M) \to \mathbb{F}$$

$$(a, b) \mapsto a \cdot b = \langle (a \cup b)[M]\rangle = \langle b, [M] \cap a\rangle$$

We want to show that capping with the fundamental class gives an isomorphism

$$H^k(M) \xrightarrow{PD} H_{n-k}(M)$$

or at least for field coefficients.

To do this, we create a geometric inverse of the map. Namely, if $N \subset M$ is smooth, closed, connected, oriented, then it has a tubular neighbourhood and we can use the Thom isomorphism.

$$\begin{tikzcd} H_n(M,\emptyset) \arrow[r] \arrow[dr] & (M|N) \arrow[d] & (V|N) \arrow[l] & (\mathcal{N},\mathcal{N}^{\#}) \arrow[l] \\ & (M|x) & & \end{tikzcd}$$

Since $H^k(N) \simeq \langle [N]^* \rangle$, the Thom isomorphism says that $H^n(\mathcal{N},\mathcal{N}^{\#}) \simeq \langle u \cup \pi^*[N]^* \rangle$. On the other hand, the fundamental class $[M]$ gives a generator $\iota_*^{-1} j_*[M]$ of $H_n(\mathcal{N},\mathcal{N}^{\#})$ by composing with the inclusion and then using excision. So we must have that

$$\langle u \cup \pi^*[N]^*, \iota_*^{-1} j_*[M] \rangle = \kappa \in \mathbb{F}^\times$$

This allows us to define a relative Thom class $u_{M/N} = \kappa^{-1} u$, which then obeys $\langle u_{M/N} \cup \pi^*[N]^*, [\mathcal{N}] = 1$, where $[\mathcal{N}]$ is the induced orientation from $[M]$.

Definition 4.21 (Geometric Poincare dual): *We now move back to M using ι, j and define*

$$pd(N) = j^*(\iota^*)^{-1} u_{M/N} \in H^{n-k}(M)$$

Proposition 4.22 (Fundamental relation):

$$PD(pd(N)) = i_*[N]$$

In other words,

$$\langle pd([N]) \cup a, [M] \rangle = \langle a, i_*[N] \rangle$$

for $a \in H^k(M)$.

Proof. We use some yoga of moving pullbacks and pushforwards, along with the normalization property.

$$\langle j^*(i^*)^{-1} u_{M/N} \cup a, [M] \rangle = \langle (i^*)^{-1} u_{M/N} \cup a, j_*[M] \rangle =$$

$$= \langle u_{M/N} \cup i^* a, (i_*)^{-1} j_*[M] \rangle = \langle u_{M/N} \cup \langle a, i_*[N] \rangle \pi^*[N]^*, i_*^{-1} j_*[M] \rangle =$$

$$= \langle a, i_*[N] \rangle$$

□

We used the fact that $\iota^*(a) = \langle a, \iota_*[N] \rangle \pi^*[N]^*$. This is true, since $V \simeq \mathcal{N}$ and $\iota^*(a) \in H^k(N) = \langle \pi^*[N]^* \rangle$, so we need to check that they both evaluate to the same thing on $[N]$. But this follows, since again we can move around pullbacks and pushforwards: $\langle \iota^* a, [N] \rangle = \langle a, \iota_*[N] \rangle$.

Recall that $a \cup b = \Delta^*(a \times b)$. The problem is, the external product is not symmetric. To make it so, we introduce the diagonal.

Let $\tilde{u}$ be the Poincare dual of the diagonal in $M \times M$, oriented via $[M] \times [M]$ (more on the homology cross product in a second). We will show the following properties of $\tilde{u}$, which will imply that PD is an isomorphism:

Proposition 4.23 (Properties of the dual of the diagonal):

- $\langle \tilde{u}, [M] \times [p] \rangle = (-1)^n$
- $\tilde{u} \cup (a \times b) = -1^{|a||b|} \tilde{u} \cup (b \times a)$
- $\langle \tilde{u}, PD(a) \times y \rangle = \pm \langle a, y \rangle$

This implies that PD is injective, but also both groups have the same dimension and it is surjective, hence it must be an isomorphism. Moreover, we have that the intersection pairing is nondegenerate.

Here, we used a homology cross product, which in our case is the dual of the cohomology external product, since we're working over a field. In particular, $\alpha \times \beta \in H_\bullet(X \times Y)$ corresponds to $\alpha \otimes \beta \in H_\bullet(X) \otimes H_\bullet(Y)$ in the sequence of dual isomorphisms. This can be characterised as

$$\langle a \times b, \alpha \times \beta \rangle = a(\alpha) b(\beta) = \langle a, \alpha \rangle \langle b, \beta \rangle$$

Using this, together with the identity $(a_1 \times a_1) \cup (b_1 \times b_2) = (a_1 \cup b_1) \times (a_2 \cup b_2)$ which follows by definition of the cross product on cohomology, we can show that

$$(z_1 \times z_2) \cap (a_1 \times a_2) = (-1)^{|a_2|(|z_1| - |a_1|)} (z_1 \cap a_1) \times (z_2 \cap a_2)$$

by applying $\langle b_1 \times b_2, - \rangle$ on both sides.

We also have that for $\alpha \in H_k(M), a \in H^k(M)$, then $\alpha \cap a = \langle a, \alpha \rangle [p]$.

Proof. We begin with the second equation. Recall that $\tilde{u} = j^*(\iota^*)^{-1} u$, so we can deal with the Thom class and then apply this. Consider the diagram

$$\begin{tikzcd}
 & M \arrow[dl, "j_\Delta"'] \arrow[d, "\Delta"] \\
V \arrow[r, "\iota'"] \arrow[d, "j'"'] & M \times M \arrow[d, "j"] \\
(V|M) \arrow[r, "\iota"] & (M \times M|\Delta)
\end{tikzcd}$$

We have

$$u \cup (\iota')^*(a \times b) = u \cup \pi^* j_\Delta^* (\iota')^*(a \times b) = u \cup \pi^* \Delta^*(a \times b) =$$
$$= u \cup \pi^*(a \cup b) = (-1)^{|a||b|} u \cup \pi^*(b \cup a) = (-1)^{|a||b|} u \cup (\iota')^*(b \times a)$$

Here, $\pi : V \to \Delta$ is the projection from the normal bundle, which has j_Δ as a homotopy inverse.

Now we move on to the first. We apply the calculation we just did:

$$\langle \tilde{u} \cup (1 \times [M]^*), [M] \times [M] \rangle = (-1)^n \langle (1 \times [M]^*) \cup \tilde{u}, [M] \times [M] \rangle =$$

$$= (-1)^n \langle \tilde{u}, ([M] \times [M]) \cap (1 \times [M]^*) \rangle = (-1)^n \langle \tilde{u}, ([M] \cap 1) \times ([M] \cap [M]^*) \rangle = (-1)^n \langle \tilde{u}, [M] \times [p] \rangle$$

But on the other hand, from 4.22, we also have

$$\langle \tilde{u} \cup (1 \times [M]^*), [M \times M] \rangle = \langle 1 \times [M]^*, [\Delta] \rangle = \langle \pi_2^*[M]^*, \Delta_*[M] \rangle = \langle [M]^*, (\pi_2)_* \Delta_*([M] \rangle = \langle [M]^*, [M] \rangle = 1$$

For the final part, we combine everything:

$$\langle \tilde{u}, PD(a) \times y \rangle = \langle \tilde{u}, ([M] \cap a) \times (y \cap 1) \rangle = \langle \tilde{u}, [M] \times y \cap a \times 1 \rangle =$$

$$= \langle (a \times 1) \cup \tilde{u}, [M] \times y \rangle = \langle (1 \times a) \cup \tilde{u}, [M] \times y \rangle = \langle \tilde{u}, ([M] \times y) \cap (1 \times a) \rangle =$$

$$= (-1)^{n|a|} \langle \tilde{u}, ([M] \cap 1) \times (y \cap a) \rangle = (-1)^{n|a|} \langle \tilde{u}, [M] \times \langle a, y \rangle [p] \rangle = (-1)^{n(n-|a|)} \langle a, y \rangle$$

□

All in all, this shows that PD is injective and that the pairing is nondegenerate.

Corollary 4.24 (Alexander duality): *Using the Poincare duality isomorphism, we can get another type of duality as follows: suppose $A \subset X$ is a submanifold in a compact manifold, and moreover has a tubular neighbourhood V, which is also its normal bundle. We get isomorphisms:*

$$H_i(X - A) \simeq H_c^{n-i}(X - A) = \varinjlim H^{n-i}(X - A, X - A - K)$$

(have not defined compactly supported cohomology unfortunately) However, the radius $1/r$ tubular neighbourhood $\frac{1}{r}V$ provides a sequence of compact subset in the complement, and then we can use excision and the homotopy equivalence $V \simeq A$ to get:

$$\varinjlim H^{n-i}(X - A, (X - A) \cap \frac{1}{r}V) \simeq \varinjlim H^{n-i}(X, \frac{1}{r}V) \simeq H^{n-i}(X, A)$$

When $X = S^n$, we can finally use the LES of a pair to get

$$H_i(S^n - A) \simeq H^{n-i-1}(A)$$

4.3.3 Gauss-Bonnet

Let's take a basis a_i of $H^\bullet(M)$, which has a dual basis b_i such that $(a_i, b_j) = \langle a_i \times b_j, [M] \rangle = \delta_{ij}$. Then we see that $\langle b_j, PD(a_i) \rangle = \langle a_i \cup b_j, [M] \rangle = \delta_{ij}$ and also $\langle a_i, PD(b_j) \rangle = \langle b_j \cup a_i, [M] \rangle = (-1)^{|a_i||b_j|}\delta_{ij}$. This shows that $PD(a_i) = b_i^*$ and $PD(b_j) = (-1)^{|a_j||b_j|}a_j^*$. We can now compute:

Corollary 4.25 (Corollary):

$$\tilde{u} = \sum (-1)^{|a_i|} a_i \times b_i$$

Proof. Just evaluate both sides on $a_i^* \times b_j^*$. □

A fundamental property of Poincare duality is that

$$pd(N_1) \cup pd(N_2) = pd(N_1 \cap N_2)$$

for transverse intersections, which comes down to the fact that the normal bundle of an intersection is the sum of the normal bundles, and the Euler class sends sums to products.

Theorem 4.26 (Gauss-Bonnet):

$$\langle e(TM), [M] \rangle = \chi(M)$$

Proof. The normal bundle of $\Delta \subset M \times M$ is just the tangent bundle. Hence,

$$\langle e(TM), [M] \rangle = \langle e(\mathcal{N}, \Delta_*[M] \rangle = \langle pd(\Delta) \cup pd(\Delta), [M \times M]$$

We can now write it in two different ways:

$$pd(\Delta) = \sum (-1)^{|a_i|} a_i \times b_i = \sum (-1)^{|b_j|} b_j \times a_j$$

When we cup them and evaluate, we get

$$\sum_{i,j} (-1)^{|a_i|} (-1)^{|b_j|} (-1)^{|b_i||b_j|} \langle a_i \cup b_j, [M] \rangle \langle b_i \cup a_j, [M] \rangle = \chi(M)$$

$\square$

5 K-theory and characteristic classes

5.1 Grassmanians and vector bundles

5.1.1 The Grassmanian as a classifying space of vector bundles

Definition 5.1 (Grassmanian):

$$\mathrm{Gr}_n(\mathbb{F}^N) = \{n\text{-dimensional vector subspaces of } \mathbb{F}^N\}$$

This has two bundles attached to it: the frame bundle, which is a principal GL_n bundle, and the tautological bundle, the associated vector bundle of the frame bundle:

$$\mathrm{Fr}_n(\mathbb{F}^N = \{(v_1, \ldots, v_n) \in (\mathbb{F}^N)^n | \text{ linearly independent}\} \subset (\mathbb{F}^N)^n$$

The map $q : \mathrm{Fr}_n \to \mathrm{Gr}_n$ sends an n-tuple to its span. This is a surjection and we topologize the Grassmanian using the quotient topology, and the frame bundle using the subset topology. Finally, we have:

$$\gamma_{\mathbb{F}}^{n,N} = \{(V, v) \in \mathrm{Gr}_n \times \mathbb{F}^N | v \in V\}$$

A neighbourhood of V in the Grassmanian consists of all the subspaces W such that $W \to \mathbb{F}^N \to V$ is an isomorphism, where the latter map is orthogonal projection.

Lemma 5.2 (Vector bundles embed into trivial bundles): *Given a compact Hausdorff space X, then any v.b. over it is a subbundle of a trivial bundle (compare with how projective modules are summands of free modules)*

Proof. Idea is to create a map

$$\phi : E \to X \times (\mathbb{F}^n)^n$$
$$e \mapsto (\pi(e), \lambda_1(\pi(e))p_1(e), \ldots)$$
$$\lambda_i \text{ are a partition of unity}$$
$$E \text{ trivialized via } U_i \times \mathbb{F}^n \simeq E|_{U_i}, (\pi(e), p_i(e)) \leftrightarrow e$$

□

Definition 5.3 (Concordance): *Two bundles E_1 and E_2 are concordant if they occur as restrictions by pulling back a bundle over $X \times I$.*

Lemma 5.4 (Concorance implies isomorphic): *If E_1 and E_2 are concordant over compact Hausdorff X, then they are isomorphic.*

Proof. See Bott-Tu. □

Proposition 5.5 (Corollary): *Pullbacks along homotopic maps are isomorphic.*

We now show we can classify vector bundles using the infinite Grassmanian and its tautological bundle, which are infinite colimits as in the diagram:

$$\begin{tikzcd} \gamma^{n,N} \arrow[r] \arrow[d] & \gamma^{n,\infty} \arrow[d] \\ \mathrm{Gr}_n(\mathbb{F}^N) \arrow[r] & \mathrm{Gr}_n(\mathbb{F}^\infty) \end{tikzcd}$$

Given a map $f : X \to \mathrm{Gr}_n(\mathbb{F}^\infty)$ we get a vector bundle on X by pulling back the tautological bundle. This does not depend on the homotopy class of f hence we get:

$$[X, \mathrm{Gr}_n(\mathbb{F}^\infty)] \to \mathrm{Vect}_n(X)/\simeq$$

Theorem 5.6 (The Grassmanian classifies vector bundles): *The above map is a bijecton*

Proof. To prove surjectivity, given E embed it into $X \times \mathbb{F}^N$ via a map ρ. Then get $X \to \mathrm{Gr_n}, x \mapsto \rho(E_x)$. Injectivity is harder: check notes. Need to use Hilbert-hotel style argument and homotope two maps $\phi_0, \phi_1 : X \to \mathrm{Gr}_n(\mathbb{F}^\infty)$ into ones using only even resp. odd coordinates. Let's denote them in the same way. Then by assumption, there is an isomorphism $\psi : \phi_0^*\gamma \simeq \phi_1^*\gamma$ which gives an isomorphism ψ_x between the vector spaces $(\phi_i^*\gamma)_x = \gamma_{\phi_i(x)} \simeq \phi_x(x)$. We can use this to create a linear homotopy, which is injective on each fiber since we modified to use only even resp. odd coordinates:

$$X \times [0,1] \to \mathrm{Gr}_n(\mathbb{F}^\infty)$$
$$(x,t) \mapsto \langle \{(1-t)v + t\psi_x(v) | v \in \phi_0(x)\} \rangle$$

□

5.1.2 The clutching construction

The suspension ΣX can be covered by two contractible cones, on which any bundle is trivial. Hence, a bundle on the suspension is determined by a map

$$C_+(X) \cap C_-(X) \simeq X \to \mathrm{GL}_n(\mathbb{F})$$

In this way, we get a bijection

$$[X, \mathrm{GL}_n(\mathbb{F})]/\pi_0 \leftrightarrow \mathrm{Vect}_n(\Sigma X)/\simeq$$

Remark : This shows that

$$[X, \mathrm{GL}_n(\mathbb{F})] \simeq [\Sigma X, \mathrm{Gr}_n(\mathbb{F}^\infty)]$$

In fact, this is because of the suspension-loop space adjunction and the fact that the loop space of the Grassmanian is homotopic to GL_n. The idea is that a loop produces a monodromy action using the tautological vector bundle! In other words, $\mathrm{Gr} = BU$.

5.2 Characteristic classes

We saw that any vector bundle $E \to X$ is classified by some map $X \to \mathrm{Gr}_n(\mathbb{F}^\infty)$. If we want to figure out whether this bundle is trivial, we need to check whether it is nullhomotopic. As a first step, which is neccesary (but not sufficient! e.g. the tangent bundle of S^5), is to see whether the map on cohomology is zero, and this is precisely what characterisrtic classes measure. There are universal spaces whose cohomology we need to understand, namely the Grassmanians, and then characteristic classes of vector bundles correspond to pullbacks along classifying maps.

5.2.1 Recollections from algebraic topology

Any R-oriented $d-$dimensional vector bundle $E \to X$ admits a Thom class $u_E \in H^d(E, E^\#; R)$ such that the following composition is an isomorphism:

$$H^i(X;R) \xrightarrow{\pi^*} H^i(E;R) \xrightarrow{u_E \cup -} H^{i+d}(E, E^\#; R)$$

The Euler class is defined by pulling back the Thom class along the zero section:

$$H^d(E, E^\#; R) \xrightarrow{q^*} H^d(E;R) \xrightarrow{s_0^*} H^d(X;R)$$

Remark: We will later need to replace $(E, E^\#)$ with the Thom pair $(\mathbf{D}(E), \mathbf{S}(E))$.

These classes are both natural and furthermore that $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$. This means that whenever a bundle has a nonzero section, it generates a line bundle inside E and hence the Euler class vanishes - in other words, the Euler class is a stable invariant.

We also have the Gysin sequence, obtained from the LES of the pair $(E, E^\#)$, the Thom isomorphism, the inverse isomorphisms s_0^*, π^* on cohomology and the projection $p : \mathbf{S}(E) \xrightarrow{i} E^\# \xrightarrow{\iota} E \xrightarrow{\pi} X$:

$$\begin{tikzcd}
\dots \arrow[r] & H^{i+d}(E,E^\#) \arrow[r, "q^*"] & H^{i+d}(E) \arrow[r, "\iota^*"] \arrow[d, bend right, "s_0^*"'] & H^{i+d}(E^\#) \arrow[r, "\delta"] \arrow[d, "i^*"] & H^{i+d+1}(E,E^\#) \arrow[r] \arrow[d, "u_E\cup -"] & \dots \\
\dots \arrow[r] & H^i(X) \arrow[r, "e(E)\cup -"'] \arrow[u, "u_E\cup -"] & H^{i+d}(X) \arrow[r, "p^*"'] \arrow[u, bend right, "\pi^*"'] & H^{i+d}(\mathbf{S}(E)) \arrow[r, "p_!"'] & H^{i+1}(X) \arrow[r] & \dots
\end{tikzcd}$$

Of special importance is the Euler class of the tautological bundle, which can be thought of a Poincare dual to minus a hyperplane. Intersecting hyperplanes corresponds to cup product, so it is natural to expect that this class generates the cohomology of $\mathbb{CP}^n$ and that is in fact the case:

Definition 5.7 (Euler class of tautological bundle): *Given a complex vector bundle $E \to X$, it is oriented and hence has an Euler class $e(E) \in H^{2d}(X;R)$ for any commutative ring R. In particular, we define:*

$$x := e(\gamma_{\mathbb{C}}^{1,n+1}) \in H^2(\mathbb{CP}^n;R)$$

Using the Gysin sequence, one can show that $H^\bullet(\mathbb{CP}^n;R) \simeq R[x]/(x^{n+1})$. Similarly, for a real vector bundle, which can only be oriented mod 2, we get $e(E) \in H^d(X;\mathbb{F}_2)$ and using the same letter

$$x := e(\gamma_{\mathbb{R}}^{1,n+1}) \in H^1(\mathbb{RP}^n;\mathbb{F}_2)$$

Again, $H^\bullet(\mathbb{RP}^n;\mathbb{F}_2) \simeq \mathbb{F}_2[x]/(x^{n+1})$

Remark: we have used the bundle $\mathcal{O}(-1)$ and thus we get that $\langle x^n, [\mathbb{CP}^n]\rangle = (-1)^n$, as will be seen later. This can be fixed by just choosing $-x$ which is the Euler class of the dual bundle $\mathcal{O}(1)$.

5.2.2 The projective bundle formula

The fact that the cohomology of projective space is generated by a single class x with a relation $x^{n+1} = 0$ can be seen as a special case to projectivizing the trivial bundle over a point. We define:

Definition 5.8 (Projectivization): *Given $E \to X$ we define a fiber bundle with fiber $\mathbb{FP}^{d-1}$ via*

$$\mathbb{P}(E) := E^\#/\mathbb{F}^\times$$

This admits a tautological line bundle

$$L_E := \{(l,v) \in \mathbb{P}(E) \times E | v \in l\}$$

$$L_E \to \mathbb{P}(E)$$

$$(l,v) \mapsto l$$

This gives us a class

$$x_E = e(L_E) \in \begin{cases} H^2(\mathbb{P}(E);R), \mathbb{F} = \mathbb{C} \\ H^1(\mathbb{P}(E);\mathbb{F}_2), \mathbb{F} = \mathbb{R} \end{cases}$$

This can also be seen as the map induced by embedding E into a trivial bundle, projectivizing and pulling back the Euler class of the tautological bundle:

$$E \to \mathbb{F}^\infty \implies \mathbb{P}(E) \to \mathbb{FP}^\infty$$

We would like to show that the cohomology of this bundle is a free $H^\bullet(X;R)$-module generated by x_E as an algebra. The relation between the x_E^i will produce the characteristic classes of E.

Theorem 5.9 (Projective bundle formula):

$$H^\bullet(X;R)\{1, x_E, \dots, x_E^{d-1}\} \simeq H^\bullet(\mathbb{P}(E);R)$$
$$\sum y_i x_E^i \mapsto \sum p^*(y_i) \cup x_E^i$$

Proof. Use Mayer-Vietoris principle, excision and the Künneth formula. Note that this is a special case of the Leray-Hirsch theorem. □

5.2.3 Chern classes

Note: the pullback map $p^* : H^\bullet(X;R) \to H^\bullet(\mathbb{P}(E);R)$ is injective by the projective bundle formula. By the projective bundle formula, the element x_E^d should be expressible in terms of the lower powers, and hence we get a relation which defines the Chern/Stiefel Whitney classes:

Definition 5.10 (Chern classes): *Given a complex vector bundle, we define $c_0(E), \dots, c_d(E)$ to be the unique classes in the cohomology of X such that $c_0(E) = 1$ and*

$$\sum (-1)^i p^*(c_i(E)) \cup x_E^{d-i} = 0 \in H^{2d}(\mathbb{P}(E);R)$$

The total Chern class is

$$c(E) = 1 + c_1(E) + \dots + c_d(E)$$

Theorem 5.11 (Properties of the Chern classes):

- *The Chern classes are invariants of the isomorphism class of the bundle E*
- $c_i(f^*E) = f^* c_i(E)$
- $$c_k(E_0 \oplus E_1) = \sum_{a+b=k} c_a(E_0) \cup c_b(E_1)$$
  or in other words $c(E_0 \oplus E_1) = c(E_0) \cup e(E_1)$.
- $c_i(E) = 0$ *if* $i \geq d$.

Proof. One proves the first two properties by considering the pullback square and then projectivizing and verifying the two things satisfy the same defining property. For the third one, the inclusions $E_1, E_2 \to E_1 \oplus E_2$ by adding a zero induce disjoint inclusions on the projective bundles (since $(0,0)$ is not an element in the projectivization). Then put $U_i = \mathbb{P}(E_1 \oplus E_2) \setminus \mathbb{P}(E_i)$. This is an open set and deformation retracts onto the other projective bundle. The idea is that each fiber looks like $\{[z_1 : \dots : z_n : w_1 : \dots : w_m] | z_i \neq 0\}$ and we can normalize the z_i to become 1's. We can thus define two classes

$$C_1 = \sum_0^n (-1)^j p^* c_j(E_1) \cup x_{E_1 \oplus E_2}^{n-j}$$
$$C_1 = \sum_0^m (-1)^j p^* c_j(E_2) \cup x_{E_1 \oplus E_2}^{m-j}$$

Note that $L_{E_1 \oplus E_2}$ restricts to the line bundles L_{E_i} over E_i and hence so does $x_{E_1 \oplus E_2}$. We thus get that C_1 restricts to 0 on $\mathbb{P}(E_1)$ by definition, and hence on U_2 and similarly C_2 on U_1. This is an open cover, so their cup product $C_1 \cup C_2$ must vanish, establishing the defining relation for the Chern classes of the Whitney sum. □

Example (Line bundles): For a line bundle $E \to X$, we have that $\rho : \mathbb{P}(E) \simeq X$ and hence that $L_E = p^* E$.

$$\begin{array}{ccc} L_E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ \mathbb{P}(E) & \xrightarrow[p]{} & X \end{array}$$

Hence, $p^* e(E) = e(L_E) = x_E$.

We now check the definition of the Chern classes:

$$0 = 1 \cup x_E - p^* c_1(E) \cup 1 \implies p^* c_1(E) = e(L_E) = p^* e(E) \implies c_1(E) = e(E)$$

Hence, for line bundles, the first Chern class is the Euler class! In particular,

$$c(\gamma_{\mathbb{C}}^{1,n+1}) = 1 + x$$

Moreover, the Chern class of a trivial bundle is 1. This implies that the Chern class is stable under adding on trivial bundles, by the sum-cup product formula.

5.2.4 Stiefel-Whitney classes

In exact analogy, we define:

Definition 5.12 (Stiefel-Whitney classes): *Given a real vector bundle, we define $w_0(E), \ldots, w_d(E)$ to be the unique classes in the cohomology of X with $\mathbb{F}_{\not\simeq}$ coefficients such that $c_0(E) = 1$ and*

$$\sum (-1)^i p^*(w_i(E)) \cup x_E^{d-i} = 0 \in H^d(\mathbb{P}(E); \mathbb{F}_2)$$

The total Stiefel-Whitney class is

$$w(E) = 1 + w_1(E) + \ldots + w_d(E)$$

These obey the same properties as the Chern classes.

Example: $w(\gamma_{\mathbb{R}}^{1,n+1}) = 1 + x$.

5.2.5 Pontrjagin classes

Definition 5.13 (Pontrjagin classes):

$$p_i(E) = (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X;R)$$

Remark: Due to complex conjugation, the odd Chern classes are 2-torsion.

5.2.6 The splitting principle

Theorem 5.14 (Splitting principle): *For a cmplex vector bundle over a compact Hausdorff space, there is a space $F(E)$ and a map $f : F(E) \to X$ such that f^*E is a sum of line bundles and f^* is injective on cohomology.*

Proof. Induction, by first using the projectivization of E. More precisely, we pull back E along the projectivized projection $\mathbb{P}(\pi)$, which contains a tautological line bundle. □

Remark: the space $F(E)$ is the flag bundle associated to E, whose fibers $F(E)_x$ consists of all chains $\emptyset \subset V_1 \subset \ldots \subset V_n = E_x$, so it is a fiber bundle with fiber equal to the flag variety.

This allows us to prove:

Theorem 5.15 (Euler classes as top Chern and Stiefel-Whitney classes): *$e(E) = c_d(E)$, E is complex of dimension d. $e(E) = w_d(E)$, E is real of dimension d*

Proof. Use the fact that it is true for line bundles, together with the splitting principle. □

Example: $c_i(\overline{E}) = (-1)^i c_i(E) = c_i(E^\vee)$.

We can use the splitting principle to show that the first Chern class of a tensor product of line bundles behaves nicely, by reducing the calculation to the universal case:

Proposition 5.16 (First Chern class of tensor product of line bundles): *For any two complex line bundles L_1, L_2 we have*

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

Proof. Firstly, consider $\mathbb{CP}^n \times \mathbb{CP}^n$ equipped with two line bundles $\pi_1^*\gamma$ and $\pi_2^*\gamma$, where π_1, π_2 are the projections and $\gamma = \gamma_{\mathbb{C}}^{1,n+1}$. We put $L = \pi_1^*\gamma \otimes \pi_2^*\gamma$. By Künneth, we know that

$$H^\bullet(\mathbb{CP}^n \times \mathbb{CP}^n;R) \simeq H^\bullet(\mathbb{CP}^n;R) \otimes H^\bullet(\mathbb{CP}^n;R) = R[x]/(x^{n+1}) \otimes R[x]/(x^{n+1})$$

We can thus express $c_1(L)$ as $Ax\otimes 1 + B1\otimes x$. We have two embeddings of $\mathbb{CP}^n$ into the product, one where the first component is fixed and one where the second one is fixed. Pulling back along L these two embeddings produces γ. If we call them ι_1, ι_2 then we have that

$$\iota_1^*(x\otimes 1) = x, \iota_2^*(1\otimes y) = x, \iota_1^*(1\otimes y) = 0, \iota_2^*(x\otimes 1) = 0$$

We thus see that $\iota_1^* c_1(L) = Ax = c_1 \iota_1^* L = c_1\gamma = x \implies A = 1$. Similarly, $B = 1$. Suggestively,

$$c_1(L) = x\otimes 1 + 1\otimes x = c_1(\pi_1^*\gamma) + c_1(\pi_2^*\gamma)$$

Now, given two complex line bundles L_1, L_2 over compact Hausdorff X, we can find $f_1, f_2 : X \to \mathbb{CP}^N$ which realizes them as pullbacks of γ. Thus,

$$L_1\otimes L_2 = (f_1\times f_2)^*(\pi_1^*\gamma\otimes\pi_2^*\gamma)$$

This shows that

$$c_1(L_1\otimes L_2) = c_1(L_1) + c_1(L_2)$$

□

Remark (First Chern class of det bundle): Using the last statement and the splitting principle, one can show that $c_1(E) = c_1(\det E)$.

Remark (Chern character): Note that the same equality can be proved using the Chern character, by comparing the degree 1 terms on both sides of the equality $\mathrm{ch}(L_1\otimes L_2) = \mathrm{ch}(L_1)\mathrm{ch}(L_2)$.

5.2.7 Calculations of tangent bundles

- First, let's explore the case of S^n. We have

  $$TS^n \oplus \mathcal{N} \simeq T\mathbb{R}^{n+1}|_{S^n}$$

  The normal bundle is trivial, and so $w(TS^n) = w(TS^n\oplus\mathcal{N}) = 1$.

- For $\mathbb{RP}^n = S^n/\mathbb{Z}/2$, we can consider the splitting above and how the antipodal map acts. On $\mathbb{R}^{n+1}$ it acts as -1 but on the normal bundle it acts as 1! By modding out, we get:

  $$T\mathbb{RP}^n \oplus \underline{\mathbb{R}} = (\gamma_{\mathbb{R}}^{1,n+1})^{\oplus n+1} \implies (1+x)^{n+1}$$

- For $\mathbb{CP}^n$ the situation is more complicated. We have γ, a subbundle of $\underline{\mathbb{C}}^{n+1}$, which has some orthogonal complement ω. We claim there is a map

  $$\phi : \mathrm{Hom}(\gamma,\omega) \to T\mathbb{CP}^n$$

  This is defined as follows (compare with 1.5.2): given $l \in \mathbb{CP}^n$, then the fiber on the LHS is $\mathrm{Hom}(l, l^\perp)$, i.e. linear maps between the line and its orthogonal complement. We want to map this fiber to the tangent space $T_l\mathbb{CP}^n$. Let $f \in \mathrm{Hom}(l, l^\perp)$. Then define

  $$Id\oplus f : l \to l\oplus l^\perp = \mathbb{C}^{n+1}$$

This deforms l in the direction of f and has image some line l_f. We can thus define a curve depending on f via

$$\alpha(t) = l_{tf} \in \mathbb{CP}^n$$

At time 0 we are just at l, so the derivative of this curve defines a tangent vector $\phi_l(f)$ Now, $\mathrm{Hom}(\gamma, \gamma) = \underline{\mathbb{C}}$ via Id and thus we get

$$\phi \oplus Id : \mathrm{Hom}(\gamma, \omega) \oplus \mathrm{Hom}(\gamma, \gamma) \to T\mathbb{CP}^n \oplus \underline{\mathbb{C}}$$

But the LHS is just $\mathrm{Hom}(\gamma, \omega \oplus \gamma) \simeq (\gamma^\vee)^{\oplus n+1} \simeq (\overline{\gamma})^{\oplus n+1}$. We conclude that

$$c(T\mathbb{CP}^n) = (1-x)^{n+1}$$

Hence, $c_1(T\mathbb{CP}^n) = (-n-1)x$, which lines up with 2.30. Also, $e(T\mathbb{CP}^n) = c_n(T\mathbb{CP}^n) = (n+1)(-x)^n = (-1)^n(n+1)x^n$. But

$$\langle e(T\mathbb{CP}^n), [\mathbb{CP}^n]\rangle = \chi(\mathbb{CP}^n) = n+1$$

since the CW structure has one cell in each even dimension $0, 2, \ldots, 2n$. This shows that $\langle x^n, [\mathbb{CP}^n]\rangle = (-1)^n$, as mentioned before. The point is that $x = c_1(\mathcal{O}(-1))$ is in some sense the "wrong" generator, and a better choice is $-x = \omega = c_1(\mathcal{O}(1))$ which actually coincides with the Fubini-Study form of projective space.

- $M = \mathbb{RP}^n \# \mathbb{RP}^n$. Check notes for this.

Remark (Grassmanians): I wonder whether the same techniques as above can tell us something about the Grassmanians as well? We have the decomposition

$$T\,\mathrm{Gr}_k(\mathbb{C}^{n+1}) \oplus \mathrm{Hom}(\gamma_{\mathbb{C}}^{k,n+1}, \gamma_{\mathbb{C}}^{k,n+1}) \simeq ((\gamma_{\mathbb{C}}^{k,n+1})^*)^{\oplus n+1}$$

The problem is that the endomorphism bundle $\gamma_{\mathbb{C}}^{k,n+1} \otimes (\gamma_{\mathbb{C}}^{k,n+1})^*$ is no longer trivial. We could try applying the Chern character, which behaves nicely with tensor products.
Also, know that there is a cover of the Grassmanian by $\binom{n+1}{k}$ contractible sets, so any $\binom{n+1}{k}$-fold products will be zero in the cohomology.
Finally, could also use Grothendieck-Riemann-Roch as well, to get an identity of the sort

$$\langle \mathrm{ch}(\gamma_{\mathbb{C}}^{k,n+1}) Td(T\,\mathrm{Gr}_k(\mathbb{C}^{n+1}))^{-1}, [\mathrm{Gr}_k(\mathbb{C}^{n+1})]\rangle = \mathrm{ch}_0\, f_!^K(\gamma_{\mathbb{C}}^{k,n+1})$$

5.2.8 Nonimmersions

$f : M \to \mathbb{R}^n$ is an immersion when its differential is everywhere injective. In this case, we get that $df : TM \to f^*T\mathbb{R}^n = \underline{\mathbb{R}^n}$ with orthogonal complement the normal bundle. Hence,

$$1 = w(\underline{\mathbb{R}^n}) = w(TM) \cup w(\mathcal{N})$$

But the normal bundle has dimension $n-d$ and hence, by formally inverting $w(TM) \in H^\bullet(M; \mathbb{F}_2)$ we get a bound on how small of a space we can immerse into. E.g.

$$w(T\mathbb{RP}^4) = 1 + x + x^4 \implies w(\mathcal{N}) = 1 + x + x^2 + x^3$$

This means that any normal bundle must be at least 3-dimensional, hence need at least 7 dimensions to immerse $\mathbb{RP}^4$.

5.2.9 Cohomology of Grassmanians

Theorem 5.17 (Cohomology of Grassmanians):

$$R[c_1, ..., c_n] \simeq H^\bullet(\mathrm{Gr}_n(\mathbb{C}^\infty); R)$$

$$\mathbb{F}_2[w_1, ..., w_n] \simeq H^\bullet(\mathrm{Gr}_n(\mathbb{R}^\infty); \mathbb{F}_2)$$

The isomorphism sends the c_i and w_i to the classes corresponding to the tautological bundle.

Proof. For the proof, one basically shows that $S(\gamma_{\mathbb{C}}^{n,\infty}) \sim \mathrm{Gr}_{n-1}(\mathbb{C}^\infty)$ and then applies the Gysin sequence. This is because

$$S(\gamma_{\mathbb{C}}^{n,\infty}) = \{(V, v) | V \text{ an n-dimensional subspace of } \mathbb{C}^\infty, v \in V, |v| = 1\}$$

We have a map

$$\begin{aligned} \Phi : S(\gamma_{\mathbb{C}}^{n,\infty}) &\to \mathrm{Gr}_{n-1}(\mathbb{C}^\infty) \\ (V, v) &\mapsto \langle v \rangle^\perp \end{aligned}$$

Conversely, we have

$$\begin{aligned} \Psi : \mathrm{Gr}_{n-1} &\to S(\gamma_{\mathbb{C}}^{n,\infty})(\mathbb{C}^\infty) \\ W &\mapsto (\mathbb{C} \oplus W, e_1) \end{aligned}$$

One composition is

$$\Phi \circ \Psi : W \mapsto \text{image of } W \text{ under the shift map } (x_1, x_2, ...) \mapsto (0, x_1, ...)$$

which is homotopic to the identity. Similarly,

$$\Psi \circ \Psi : (V, v) \mapsto (\mathbb{C} \oplus \langle v \rangle^\perp, e_1)$$

which is homotopic using Hilbert trick and interpolating between v and e_1.

Now, we have

$$\begin{tikzcd} \mathrm{Gr}_{n-1}(\mathbb{C}^\infty) \arrow[r, "\sim"] \arrow[dr, "inc"'] & S(\gamma_{\mathbb{C}}^{n,\infty}) \arrow[d, "p"] \\ & \mathrm{Gr}_n(\mathbb{C}^\infty) \end{tikzcd}$$

We get the Gysin LES:

$$... \to H^i(\mathrm{Gr}_n) \xrightarrow{-\cup c_n(\gamma)} H^{i+2n}(\mathrm{Gr}_n) \xrightarrow{inc^*} H^{i+2n}(\mathrm{Gr}_{n-1}) \to H^{i+1}(\mathrm{Gr}_n) \to ...$$

One shows that inc^* is surjective and cupping with the top Chern class, i.e. the Euler class of the tautological bundle, is injective, and hence we get an exact triangle

$$\begin{tikzcd} & H^\bullet(\mathrm{Gr}_n) \arrow[dr, "e(\gamma)\cup -"] & \\ H^{\bullet+2n}(\mathrm{Gr}_{n-1})) \arrow[ur, "inc_![1]"] & & H^{\bullet+2n}(\mathrm{Gr}_n)) \arrow[ll, "inc^*"] \end{tikzcd}$$

which gives a SES of rings

$$0 \to H^\bullet(\mathrm{Gr}_n) \xrightarrow{c_n(\gamma)\cup -} H^\bullet(\mathrm{Gr}_n) \to H^\bullet(\mathrm{Gr}_{n-1}) \to 0$$

By induction and the five lemma, one can conclude the result in the theorem. □

Remark (Relationship with symmetric polynomials): Consider the classifying map $f : (\mathbb{CP}^\infty)^n \to \mathrm{Gr}_n(\mathbb{C}^\infty)$ of the n-fold Cartesian product of the tautological line bundle. Then if γ_i denotes the first Chern class of the subbundle with all except the i-th product entry equal to the tautological bundle, then

$$R[c_1, ..., c_n] \simeq H^\bullet(\mathrm{Gr}_n(\mathbb{C}^\infty)) \xrightarrow{f^*} H^\bullet((\mathbb{CP}^\infty)^n) \simeq R[\gamma_1, ..., \gamma_n]$$
$$f^*(c_i) = \sigma_i(\gamma_1, ..., \gamma_n)$$

5.2.10 The orientable Grassmanian

Definition 5.18 (Orientable Grassmanian):

$$\begin{tikzcd} \widetilde{\mathrm{Gr}_n(\mathbb{R}^\infty)} = \{(V,\omega), \omega \textit{ an orientation}\} \arrow[d, "p", "2:1"'] \\ \mathrm{Gr}_n(\mathbb{R}^\infty) \end{tikzcd}$$

An orientation is a choice of unit vector in $\bigwedge^n V$, so we can canonically identify the orientable Grassmanian with $\mathbf{S}(\bigwedge^n \gamma_{\mathbb{R}}^{n,\infty})$.

Pulling back the tautological bundle along p we get an oriented bundle over the orientable Grassmanian. The nonorientability of the usual Grassmanian didn't allow us to use different coefficient rings except $\mathbb{F}_2$. However, for the orientable cover, the same is not true and we have:

Theorem 5.19 (Cohomology of orientable Grassmanian): *If 2 is a unit in R and $k > 0$ then*

$$R[e, p_1, ..., p_{k-1}] \simeq H^\bullet(\widetilde{\mathrm{Gr}_{2k}}(\mathbb{R}^\infty); R)$$
$$R[p_1, ..., p_{k-1}] \simeq H^\bullet(\widetilde{\mathrm{Gr}_{2k-1}}(\mathbb{R}^\infty); R)$$
$$\mathbb{F}_2[w_2, ..., w_n] \simeq H^\bullet(\widetilde{\mathrm{Gr}_n}(\mathbb{R}^\infty); \mathbb{F}_2)$$

The proof idea is the same, using the Gysin sequence. The induction, however, begins with $\widetilde{\mathrm{Gr}_1}(\mathbb{R}^\infty) \simeq \mathbf{S}(\bigwedge^1 \gamma_{\mathbb{R}}^{1,\infty}) \simeq S^\infty \simeq *$, i.e. the sphere bundle of the tautological bundle over $\mathbb{RP}^\infty$ is just the two-fold cover given by S^∞. Note that we need 2 to be a unit to get $2e(\tilde{\gamma}_{\mathbb{R}}^{2k-1,\infty}) = 0 \implies e(\tilde{\gamma}_{\mathbb{R}}^{2k-1,\infty}) = 0$. Note also that the top Pontryagin class is the square of the Euler class.

To get back to the cohomology of the unoriented Grassmanian, we need the following:

Lemma 5.20 (Lemma): *If $\tilde{X} \to X$ is an m-fold covering map and m is invertible in R, then the induced map on cohomology is injective.*

Proof. The proof goes by noting that any singular simplex $\sigma : \Delta^p \to X$ has m different lifts to $\tilde{X}$ and hence we get a map

$$C_\bullet(X) \xrightarrow{\tau} C_\bullet(\tilde{X})$$
$$\sigma \mapsto \sum \tilde{\sigma}$$

Hence, $p_* \circ \tau = m.\mathbf{1}$. On cochains, since m is invertible, this becomes a onesided inverse for p^*. ☐

As a corollary, we get:

Proposition 5.21 (Cohomology of real Grassmanians): *If 2 is a unit in R, then*

$$R[p_1, ..., p_{k-1}] \simeq H^\bullet(\mathrm{Gr}_{2k}(\mathbb{R}^\infty); R)$$
$$R[p_1, ..., p_{k-1}] \simeq H^\bullet(\mathrm{Gr}_{2k-1}(\mathbb{R}^\infty); R)$$

Proof. We use the above lemma to see that the cohomology of Gr certainly injects in those rings. Then we use the following trick: the orientable Grassmanian has an involution which reverse the orientation and which on cohomology acts as the identity on the Pontrjagin classes, but as -1 on e. ☐

5.2.11 Further facts about characteristic classes, obstruction theory, etc.

Proposition 5.22 (Chern class as a homomorphism): *If X has the homotopy type of a CW complex, then the map*

$$c_1 : \mathrm{Vect}^1_{\mathbb{C}}(X) \to H^2(X; \mathbb{Z})$$

from line bundles to second cohomology is an isomorphism (note that this is not the same as 2.24, as there we are talking about holomorphic line bundles, and here only about topological complex line bundles). The same is true for w_1, real line bundles and $H^1(X; \mathbb{Z}_2)$.

Proof. We have shown that $\mathbb{CP}^\infty$ classifies line bundles up to isomorphism, so $[X, \mathbb{CP}^\infty] \simeq \mathrm{Vect}^1_{\mathbb{C}}(X)$. On the other hand, $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ space, so it classifies cohomology: $[X, \mathbb{CP}^\infty] \simeq H^2(X; \mathbb{Z})$, given by pulling back the universal class x. We thus have the factorization of isomorphisms

$$[X, \mathbb{CP}^\infty] \to \mathrm{Vect}^1_{\mathbb{C}}(X) \to H^2(X;\mathbb{Z})$$
$$f \mapsto f^*(\gamma) \mapsto c_1(f^*\gamma) = f^*(x)$$

□

w_1 as obstruction to orientablity, Euler class as obstruction to sections, Pontrjagin classes and how they relate to Chern and Stiefel-Whitney classes, obstruction theory using cell structures..

5.3 K-theory

Definition 5.23 (K-theory):

$$K^0(X) = \text{ Grothendieck group of } Vect(X)$$

The identity is the trivial bundle, and the operation is sum. An element will be writen by $E - F$. Two such elements are equal if there is a C such that $E \oplus F' \oplus C \simeq E' \oplus F \oplus C$. This is functorial, and descends to the homotopy category, due to the concordance theorem. It also has a ring structure, defined by the tensor product and with unit the trivial line bundle over X. Finally, we also have a cup product:

$$K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$$
$$x \otimes y \mapsto \pi_X^* x \otimes \pi_Y^* y$$

This is a ring homomorphism if we equip the tensor product with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$.

Example: $K^0(pt) \simeq \mathbb{Z}$

Choosing a basepoint, have a rank function which takes a vector bundle and tells us its rank at x_0. This extends to K^0 and its kernel is the reduced K theory, and this map is split by $n \mapsto \pm\underline{\mathbb{C}^{|n|}}_X$. Hence,

$$K^0(X) = \tilde{K}^0(X) \oplus \mathbb{Z}$$

Lemma 5.24 (Lemma): *If X is compact Hausdorff, then every element of K-theory can be represented as*

$$E - \underline{\mathbb{C}^n}_X$$

For reduced K-theory, the n is $dim E_{x_0}$.

We have an alternate description of reduced K-theory, namely as the set of vector bundles up to stable isomorphism:

$$E \sim E' \iff E \oplus \underline{\mathbb{C}^n}_X = E' \oplus \underline{\mathbb{C}^m}_X$$

Another way to think about it is as the kernel ideal of the pullback $\iota : \{x_0\} \to X$.

Remark (Alternative definition of K-theory): One might as well define ordinary K-theory as the reduced K-theory of the 1-point compactification: $K^i(X) = \tilde{K}^i(X_+)$

Example (K-theory of the circle): $\tilde{K}^0(S^1) = 0$, since by clutching, v.b.s over S^1 correspond to homotopy classes of maps

$$[S^0, GL(n, \mathbb{C})] = *$$

since it is path-connected.

5.3.1 K-theory of $\mathbb{CP}^1$ and the fundamental product theorem

Proposition 5.25 (Generating relation): *Let $H = [\overline{\gamma_{\mathbb{C}}^{1,2}}] \in K^0(\mathbb{CP}^1)$. Then $H + H = H^2 + 1$ i.e. $(H-1)^2 = 0$.*

Proof. We will use the fact that $\mathbb{CP}^1 \simeq S^2 = \Sigma S^1$ and the clutching construction. Let

$$C_0 = \{[1:z] | |z| \leq 1\}$$
$$C_1 = C_0 = \{[w:1] | |w| \leq 1\}$$
$$C_0 \cap C_1 = C_0 = \{[1:z] | |z| = 1\} = S^1$$

Over each bit, the tautological line bundle is trivialized using

$$C_0 \times \mathbb{C} \to \gamma|_{C_0}, ([z:1], \lambda) \mapsto ([z:1], \lambda(z,1))$$
$$C_1 \times \mathbb{C} \to \gamma|_{C_1}, ([1:w], \lambda) \mapsto ([1:w], \lambda(1,w))$$

Hence, on the intersection, the transition function is multiplication by z, in other words just the inclusion

$$S^1 \to \mathbb{C}^\times$$

Now we compare clutching functions:

$$\gamma \oplus \gamma \leftrightarrow \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$
$$(\gamma \otimes \gamma) \oplus \mathbb{C} \leftrightarrow \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

These are in fact homotopic, hence give the same vector bundle. $\square$

Note that, due to characteristic class reasons, $H \neq 1$. This gives a ring homomorphism

$$\phi : \mathbb{Z}[H]/(H-1)^2 \to K^0(\mathbb{CP}^1)$$

We will later prove the fundamental product theorem, which says that

$$K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \simeq K^0(X \times \mathbb{CP}^1)$$

5.3.2 K-theory as a cohomology theory and the Bott periodicity theorem

Lemma 5.26 (Technical lemma): *Since we're working over compact Hausdorff spaces, we need the following fact: if $E \to X$, $A \subset X$ closed such that $E|_A$ is trivial, then it is also trivial in some open neighbourhood of A.*

5.3.2.1 LES of a pair

Proposition 5.27 (Relative K-theory): *Let A be a closed subset of X. Then*

$$(A, x_0) \xrightarrow{\iota} (X, x_0) \xrightarrow{q} (X/A, A/A)$$

induces a map on K-theory which is exact in the middle:

$$\tilde{K}^0(X, A) := \tilde{K}^0(X/A) \xrightarrow{q^*} \tilde{K}^0(X) \xrightarrow{\iota^*} \tilde{K}^0(A)$$

Proof. The composition factors through the constant map to a point, and hence is 0 on reduced K-theory. Conversely, given E over X such that $\iota^* E$ is stably trivial, suppose WLOG that $E|_A \to A$ is trivial by adding on a trivial bundle. Hence, we have an isomorphism $h : E|_A \simeq A \times \mathbb{C}^n$.We can create a vector bundle $E/h \to X/A$ by defining $(E/h)_x = E_x$ for $x \notin A$ and every fiber over any element of a is squished together, $h^{-1}(a, v) \sim h^{-1}(a', v)$. This is locally trivial by the previous lemma. We thus have a pullback

$$\begin{array}{ccc} E & \longrightarrow & E/h \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{q} & X/A \end{array}$$

which realizes E as $q^*(E/h)$. ☐

Given $f : X \to Y$, recall the construction of the mapping cylinder and mapping cone:

$$M_f = \frac{X \times [0,1] \coprod Y}{(x, 1) \sim f(x)}, C_f = \frac{M_f}{X \times \{0\}}$$

This factorizes f as an inclusion and deformation retraction:

$$X \to M_f \to Y$$

Lemma 5.28 (Excision for K-theory): *Given $A \subset X$ closed and contracible inside compact Hausdorff X, then the map $c : X \to X/A$ induces an isomorphism on reduced K-theory.*

Proof. The idea is that $E|_A$ is trivial, hence have an isomorphism $h : E|_A \simeq A \times \mathbb{C}^n$. As before, $q^*(E/h) = E$ so c^* is surjective. We also need to show it is injective, i.e. E/h does not depend on the choice of trivialization h. Here, we use contractibility of A: if we have two trivs h_1, h_2, they define

$$h_0 h_1^{-1} : A \times \mathbb{C}^n \to A \times \mathbb{C}^n, (a, v) \mapsto (a, g(a)v), g : A \to \mathrm{GL}(n, \mathbb{C})$$

But A is contracible so $g \sim I$ the constant map via a homotopy $H : A \times I \to \mathrm{GL}(n, \mathbb{C})$. We get a homotopy of trivializations

$$E|_A \times I \to A \times I \times \mathbb{C}^n$$

in other words get a trivialization of $E|_A \times I \to A \times I$ that interpolates between h_0 and h_1. Hence, we can construct $E|_A \times I/H$ that gives a concordance between E/h_0 and E/h_1. □

Proposition 5.29 (Corollary): *Given a closed subspace $A \subset X$, the collapse map $c : C_\iota \to X/A$ induces an isomorphism on reduced K-theory.*

Proof. The collapse map kills off the cone on A, which is a closed contractible subset of C_ι. By the previous lemma, we're done. □

So everytime we have an inclusion of a closed subspace in a larger space, the cone of the inclusion is the same, K-theoretically, as the quotient.

Let's see what happens when we iterate the mapping cone construction, which should give a long exact sequence on K-theory on the top row. We have the following situation: the approx arrows are isomorphism on $\tilde{K}^0$ and all the horizontal arrows are inclusions.

$$\begin{tikzcd}
A \arrow[r, "\iota_0"] & X \arrow[r, "\iota_1"] \arrow[d] & C_{\iota_0} \arrow[r, "\iota_2"] \arrow[d] \arrow[dl, "\sim"] & C_{\iota_1} \arrow[r, "\iota_3"] \arrow[d] \arrow[dl, "\sim"] & C_{\iota_2} \arrow[r, "\iota_4"] \arrow[d] \arrow[dl, "\sim"] & C_{\iota_3} \arrow[dl, "\sim"] \\
& X/A & C_{\iota_0}/X \sim \Sigma A & C_{\iota_1}/C_{\iota_0} \sim \Sigma X & C_{\iota_2}/C_{\iota_1} \sim \Sigma C_{\iota_0} &
\end{tikzcd}$$

We know that $X/A \approx C_{\iota_0}$ by the lemma. However, we want to show the same is true when we take suspensions, so that $\Sigma C_{\iota_0} \approx \Sigma X/A$ and we can then get the long exact sequence. However, when we take the suspension of the collapse map $C_{\iota_0} \to X/A$, we are basically collapsing on every level separately, hence we have collapsed $\Sigma C(A)$ to a contractible line. We can further collapse this contractible set, which is an isomorphism on reduced K-theory by the excision property. On the other hand, we could have just contracted the whole $\Sigma C(A)$ from the get go, which is contractible, so induces an iso on K-theory. In other words, we have the following commuting diagram:

$$\begin{tikzcd}
\Sigma C_{\iota_0} \arrow[r, "\Sigma c"] \arrow[rr, bend right, "\sim"'] & \Sigma(X/A) \arrow[r] & \Sigma(X/A)/\{x\} \times I
\end{tikzcd}$$

In other words, suspending the collapse map, then killing of the basepoint times the interval is the same as collapsing the contractible cone on A, so Σc induces an isomorphism on $\tilde{K}^0$. With this information, we get a long exact sequence, where q denotes quotient map and c collapse:

$$\begin{tikzcd}
\tilde{K}^0(A) & \tilde{K}^0(X) \arrow[l, "\iota^*"'] & \tilde{K}^0(X/A) \arrow[l, "q^*"'] \arrow[dr, "c^*"'] & & \tilde{K}^0(\Sigma A) \arrow[ll, "\partial"'] \arrow[dl, "q^*"] & \tilde{K}^0(\Sigma X) \arrow[l, "\iota^*"'] & \tilde{K}^0(\Sigma(X/A)) \arrow[l, "q^*"'] \\
& & & \tilde{K}^0(C_{\iota_0}) & & &
\end{tikzcd}$$

We see that $\partial = (c^*)^{-1} q^*$.

Exactness at this stage can be verified as follows: suppose $[F] \in \ker q^* \subset \tilde{K}^0(X/A)$. Hence,

$$0 = q^*[F] = \iota_1^* c^*[F]$$

since the quotient map $X \xrightarrow{q} X/A$ factors as $X \xrightarrow{\iota_0} C_{\iota_0} \xrightarrow{c} X/A$.

Therefore, we can use that

$$X \xrightarrow{\iota_1} C_{\iota_0} \xrightarrow{q} C_{\iota_0}/X = \Sigma A$$

becomes exact on K-theory to see that

$$\iota_1^* c^*[F] = 0 \implies c^*[F] \in \operatorname{im} q^* \implies [F] \in \operatorname{im} \partial$$

I guess C_{ι_2} is homotopic to ΣX, so a lot of this was irrelevant? Can just use the cone iteration to define the boundary map, and then glue.

No! Confusion is that one need to use minus the suspension, i.e. the induced map that is given by $t \mapsto 1 - t$ or something like this, need to double check! Really, the cone iteration is exact on the level of K-theory but it tells us something about concrete spaces by the lemma.

5.3.2.2 External product on reduced K-theory

We defined the box product

$$K^0(X) \otimes K^0(Y) \xrightarrow{\boxtimes} K^0(X \times Y)$$

by pulling back along the two projections and tensoring. However, a different approach will be used for reduced K-theory, where all of $X \vee Y$ can serve as a basepoint, so we would rather work with $X \wedge Y = X \times Y / X \vee Y$.

Lemma 5.30 (K-theory of a wedge): *$\iota_X^* \oplus \iota_Y^* : \tilde{K}^{-i}(X \vee Y) \simeq \tilde{K}^{-i}(X) \oplus \tilde{K}^{-i}(Y)$, for all $i \geq 0$*

Proof. Consider the LES of the pair $(X \vee Y, X)$:

$$\tilde{K}^0(X) \xleftarrow{\iota_X^*} \tilde{K}^0(X \vee Y) \xleftarrow{r_Y^*} \tilde{K}^0(Y) \xleftarrow{\partial} \dots$$

We have that $r \circ \iota = 1$, so r is injective and ι is surjective. (r_Y resp. r_X collapses one of the spaces to a point). This shows that the following is exact, split by ι_Y:

$$0 \longleftarrow \tilde{K}^{-i}(X) \xleftarrow{\iota_X^*} \tilde{K}^{-i}(X \vee Y) \xleftarrow{r_Y^*} \tilde{K}^{-i}(Y) \longleftarrow 0$$

$\square$

Now consider the LES of a pair $(X \times Y, X \vee Y)$, together with the isomorphism from the above lemma:

$$\tilde{K}^{-i}(X) \oplus \tilde{K}^{-i}(Y) \xleftarrow{\iota_X^* \oplus \iota_Y^*} \tilde{K}^{-i}(X \vee Y) \xleftarrow{i^*} \tilde{K}^{-i}(X \times Y) \xleftarrow{q^*} \tilde{K}^{-i}(X \wedge Y) \longleftarrow \ldots$$

We see that $(\iota_X^* \oplus \iota_Y^*) \circ i^*$ is surjective, split by the two projections $\pi_X^* \oplus \pi_Y^*$. Hence, we get a decomposition

$$\tilde{K}^{-i}(X \times Y) \simeq \tilde{K}^{-i}(X \wedge Y) \oplus \tilde{K}^{-i}(X) \oplus \tilde{K}^{-i}(Y)$$

We now see that if x, y are classes in the reduced K theory of X, Y, then the cup product $\pi_X^* x \otimes \pi_Y^* y \in \tilde{K}^0(X \times Y)$ will vanish when restricted to $X \times \{y_0\}, \{x_0\} \times Y$, i.e vanishes when i^* is applied, and hence must land inside $\tilde{K}^0(X \wedge Y)$! This allows us to define:

Definition 5.31 (External product on reduced K-theory): *Given $x, y \in \tilde{K}^0(X), \tilde{K}^0(Y)$, their cup product lies in the summand of $\tilde{K}^0(X \times Y)$ corresponding to $\tilde{K}^0(X \wedge Y)$ and we can then define*

$$- \boxtimes - : \tilde{K}^0(X) \otimes \tilde{K}^0(Y) \to \tilde{K}^0(X \wedge Y)$$

One recovers the internal product by pulling back the diagonal.

Example (n-fold products are zero when there is a cover by contractible sets): Let A, B be contractible closed sets which cover X. Then $X \to X/A, X \to X/B$ induce isomorphisms on K-theory and we have that the internal product corresponds to the external product

$$\tilde{K}^0(X/A) \otimes \tilde{K}^0(Y/B) \to \tilde{K}^0(X/A \wedge X/B)$$

However, note that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \wedge X \arrow[d] \\ * = X/(A \cup B) \arrow[r] & X/A \wedge X/B \end{tikzcd}$$

Hence, the map factors through the reduced K-theory of a point and must be zero. This fact can be generalized to arbitrary covers (note that the same thing holds for cup products in ordinary singular cohomology)

5.3.2.3 The graded multiplication and the Bott map

We have been working throughout with the unreduced suspension ΣX. However, note that the comparison map

$$c : \Sigma X \to S^1 \wedge X$$

collapses $[0,1]\times\{x_0\}$, which is contractible, hence induces an isomorphism on K-theory. Similarly for the n-fold suspension $c^* : \tilde{K}^0(S^n \wedge X) \simeq \tilde{K}^0(\Sigma^n X)$ and this allows us to define the Bott map, which we claim is an isomorphism:

Theorem 5.32 (Bott isomorphism): *Recall that $K^0(\mathbb{CP}^1) \simeq \mathbb{Z}[H]/(H-1)^2$ by the fundamental product theorem. Then $\tilde{K}^0(\mathbb{CP}^1) \simeq \mathbb{Z}[H-1]$ and hence we put*

$$\begin{aligned}\beta : \tilde{K}^0(X) \to \tilde{K}^0(X \wedge \mathbb{CP}^1) \simeq \tilde{K}^0(\Sigma^2 X) = \tilde{K}^{-2}(X)\\ x \mapsto (H-1)\boxtimes x\end{aligned}$$

This is an isomorphism.

This will follow by the Fundamental Product theorem and allows us to make the LES into an exact hexagon:

$$\begin{tikzcd}
\tilde{K}^0(A) \arrow[d, "\partial"'] & \tilde{K}^0(X) \arrow[l, "i^*"'] & \tilde{K}^0(X/A) \arrow[l, "q^*"'] \\
\tilde{K}^{-1}(X/A) \arrow[r, "q^*"'] & \tilde{K}^{-1}(X) \arrow[r, "i^*"'] & \tilde{K}^{-1}(A) \arrow[u, "\partial"']
\end{tikzcd}$$

Example (K-theory of spheres): Since all spheres are iterated suspensions of S^0, we only need to know $K^0(S^0)$ and $K^{-1}(S^0$. However, $K^0(S^0) \simeq \mathbb{Z}\oplus\mathbb{Z}$, whereas $K^{-1}(S^0) = \tilde{K}^0(S^1) \simeq 0$ by looking at the clutching functions. Hence,

$$\tilde{K}^i(S^{2n}) = \begin{cases} \mathbb{Z}, i = 0 \\ 0, i = -1 \end{cases}$$

$$\tilde{K}^i(S^{2n+1}) = \begin{cases} 0, i = 0 \\ \mathbb{Z}, i = -1 \end{cases}$$

In particular, the degree d map $f : S^1 \to S^1$ induces an automorphism f^* on $\tilde{K}^{-1}(S^1)$ which is by definition the automorphism given by $(\Sigma f)^*$ on $\tilde{K}^0(S^2)$. Since this is generated by the class $[H-1]$, we have a relation $\Sigma f^*[H-1] = k[H-1]$ as reduced K-theory classes. In particular, after taking conjugates this gives the relation

$$\Sigma f^*\gamma \oplus \underline{\mathbb{C}}^N \simeq \gamma^{\oplus k} \oplus \underline{\mathbb{C}}^{N+1-k}$$

Taking Chern classes, we get that

$$\Sigma f^*(1+x) = (1+x)^k = 1 + kx \in H^\bullet(S^2;\mathbb{Z})$$

and hence, since f has degree d, $k = d$.

Now we move on to the graded ring structure of K-theory. The diagonal map $\Delta : X \to X \wedge X$

induces maps

$$S^{i+j} \wedge \Delta : S^{i+j} \wedge X \to S^{i+j} \wedge X \wedge X \simeq (S^i \wedge X) \wedge (S^j \wedge X)$$

which on reduced K-theory gives

$$\tilde{K}^0(S^i \wedge X) \otimes \tilde{K}^0(S^j \wedge X) \xrightarrow{\boxtimes} \tilde{K}^0(S^i \wedge X \wedge S^j \wedge X) \xrightarrow{(S^{i+j} \wedge \Delta)^*} \tilde{K}^0(S^{i+j} \wedge X)$$

Definition 5.33 (Graded multiplication on K-theory): *The resulting map, using the isomorphisms c^* between the reduced and unreduced suspension, together with an additional sign $(-1)^{ij}$ due to some permutations happening with the spheres, defines a product*

$$\tilde{K}^{-i}(X) \otimes \tilde{K}^{-i}(Y) \xrightarrow{\otimes} \tilde{K}^{-i-j}(X)$$

Remark: Just like the cup product, this is graded commutative. The reason for inserting the signs is justified here: `https://mathoverflow.net/questions/441484/the-graded-multiplication-on-topological-`

5.3.2.4 The Mayer-Vietoris sequence

Suppose $X = A \cup B$ where A, B are closed subsets. Thus $X/A \simeq B/A \cap B$ and we have a map of LES's between (X, A) and $(B, A \cap B)$ as follows:

$$\begin{tikzcd}
\dots & \tilde{K}^{-1}(X/A) \arrow[l] \arrow[d, "\simeq"] & K^0(A) \arrow[l, "\partial"'] \arrow[d, "j_A^*"] & K^0(X) \arrow[l, "i_A^*"'] \arrow[d, "i_B^*"] & \tilde{K}^0(X/A) \arrow[l, "q_A^*"'] \arrow[d, "\simeq"] & K^{-1}(A) \arrow[l, "\partial"'] \arrow[d, "j_A^*"] & \dots \arrow[l] \\
\dots & \tilde{K}^{-1}(B/A \cap B) \arrow[l] & K^0(A \cap B) \arrow[l, "\partial"] & K^0(B) \arrow[l, "j_B^*"] & \tilde{K}^0(B/A \cap B) \arrow[l, "q_{A\cap B}^*"] & K^{-1}(A \cap B) \arrow[l, "\partial"] & \dots \arrow[l]
\end{tikzcd}$$

By an exercise in homological algebra (see Hatcher section on Mayer-Vietoris), this induces a LES

$$\begin{tikzcd}
K^0(A \cap B) \arrow[d, "\partial'"'] & K^0(A) \oplus K^0(B) \arrow[l, "j_A^* - j_B^*"'] & K^0(X) \arrow[l, "i_A^* \oplus i_B^*"'] \\
K^{-1}(X) \arrow[r, "i_A^* \oplus i_B^*"'] & K^{-1}(A) \oplus K^{-1}(B) \arrow[r, "j_A^* - j_B^*"'] & K^{-1}(A \cap B) \arrow[u, "\partial'"']
\end{tikzcd}$$

where ∂' is the composition of ∂ with the isomorphism and then q_A^*. This produces a Mayer-Vietoris sequence for K-theory. Note that all of the ingredients we needed were the naturality of a LES of a pair, from which it fell out.

5.3.3 Classifying spaces for K-theory

This section serves as an additional discussion of Bott periodicity and some extra topics.

5.3.3.1 Defining K-theory

Structure of the talk:

- define complex topological K-theory, prove Serre-Swan to connect it to functional analysis
- Find classifying spaces for K-theory, namely $\mathbb{Z} \times BU$ and $\mathcal{F}$ using different techniques. Explain how they define a cohomology theory, using Ω-spectra.
- Prove Bott periodicity and explain what it has to do with the stable homotopy groups of U; one proof using McDuff's quasifibration, and the other using the index of elliptic operators

The set of complex vector bundles over a compact Hausdorff X forms an abelian monoid under the Whitney sum operation. The Grothendieck completion is thus a commutative ring.

5.3.3.2 Serre-Swan, or how to define K-theory using functional analysis

Theorem 5.34 (Serre-Swan): *There is an equivalence of categories between vector bundles over a compact Hausdorff X and finitely generated projective $C(X)$-modules.*

Proof. Given $E \to X$, associate $\Gamma(E, X)$, the space of sections, which is a f.g. projective $C(X)$-module, since E embeds into a trivial bundle for compact Hausdorff X. Conversely, given $\mathcal{E}$ a f.g. projective module, we can create

$$E = \coprod_p E \otimes_{C(X)^p} \mathbb{C}$$

where the action of $C(X)^p$ on $\mathbb{C}$ is given by the evaluation map: $f \cdot z = f(p)z$. Note that the evaluation maps form the characters i.e. maximal ideals inside $C(X)$:

$$\Phi_{C(X)} = \mathrm{Hom}_{BA}(C(X), \mathbb{C}) = \{ev_p | p \in X\}$$

$\square$

This tells us that the K-theory of the space X is the same as the K-theory of the ring $C(X)$.

5.3.3.3 Using algebraic topology

We have a map

$$\mathrm{Vect}_n(X) \to \tilde{K}(X)$$
$$E \mapsto E - [n]$$

This is compatible with the stabilization map $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X), E \mapsto E \oplus \underline{\mathbb{C}}$. Hence, we this induces a map $\varinjlim \mathrm{Vect}_n(X) \to \tilde{K}(X)$ which is in fact an isomorphism.

Now, recall that the infinite Grassmanian classifies vector bundles:

$$\mathrm{Vect}_n(X) \simeq [X, \mathrm{Gr}_n(\mathbb{C}^\infty)]$$

In other words, $\mathrm{Gr}_n = BU(n)$ and (if one is careful about point-set issues) $\mathrm{Gr}_\infty(\mathbb{C}^\infty) = BU(\infty)$. This is true for the following reasons:

Consider the fibration

$$\begin{tikzcd} \mathrm{GL}_n(\mathbb{C}) \arrow[r] & \mathrm{Fr}_n(\mathbb{C}^\infty) \arrow[d] \\ & \mathrm{Gr}_n(\mathbb{C}^\infty) \end{tikzcd}$$

One can show that the frame bundle (called the Stiefel manifold, a principal bundle) is contractible, so by Hatcher, proposition 4.66, we have a homotopy equivalence $U(n) \simeq \Omega \mathrm{Gr}_n(\mathbb{C}^\infty)$.

More generally, for G a topological group there is a fiber bundle

$$\begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG \end{tikzcd}$$

such that EG is contractible, and using the same proposition we get that $\Omega BG \simeq G$.

In any case, we get the fact that

$$\tilde{K}(X) \simeq \varinjlim \mathrm{Vect}_n(X) \simeq \varinjlim [X, \mathrm{Gr}_n(\mathbb{C}^\infty] \simeq [X, \mathrm{Gr}] \simeq [X, BU]$$

We can also show that $K(X) = [X, \mathbb{Z} \times BU]$ and hence we have found a classifying space for K-theory - the classifying space of the infinite unitary group!

5.3.3.4 Omega-spectra, The Puppe (cofibration) sequence and the triangulated structure on topological spaces

For a CW complex X one can obtain a series of maps

$$A \to X \to X/A \to \Sigma X \to \Sigma X \to \Sigma(X/A) \to \ldots$$

Importantly, this becomes exact when one applies the contravariant functor $[-, K]$. Suppose we have an omega-spectrum $K_n = \Omega K_{n+1}$. Then we have basically defined a cohomology theory:

$$
\begin{tikzcd}
 & & & {[A,K_n]} \arrow[d, phantom, "\wr"] & {[X,K_n]} \arrow[l] \arrow[d, phantom, "\wr"] & {[X/A,K_n]} \arrow[l] \arrow[d, phantom, "\wr"] \\
 & & & {[A,\Omega K_{n+1}]} \arrow[d, phantom, "\wr"] & {[X,\Omega K_{n+1}]} \arrow[l] \arrow[d, phantom, "\wr"] & {[X/A,\Omega K_{n+1}]} \arrow[l] \arrow[d, phantom, "\wr"] \\
{[A,K_{n+1}]} & {[X,K_{n+1}]} \arrow[l] & {[X/A,K_{n+1}]} \arrow[l] & {[\Sigma A,K_{n+1}]} \arrow[l] & {[\Sigma X,K_{n+1}]} \arrow[l] & {[\Sigma(X/A),K_{n+1}]} \arrow[l]
\end{tikzcd}
$$

We have seen that $\tilde{K}(X) = [X, BU]$. Thus, we can use this classifying space to create an Omega-spectrum, giving us the full cohomology theory of K-theory. Since, $\Omega BU \simeq U$, Bott periodicity then is equivalent to the statement that $\Omega^2 U \simeq U$, giving us full control of the homotopy groups of the unitary group!

5.3.3.5 Using Fredholm operators

Idea: given a map of vector spaces T, can form

$$0 \to \ker T \to H \to H' \to \operatorname{coker} T \to 0$$

By linear algebra, the dimensions add up and so :

$$\ker T \oplus H' = H \oplus \operatorname{coker} T$$

So, if we look at their dimensions, we can identify $H' - H$ with $\ker T - \operatorname{coker} T$ in the K-theory of a point. We can extend this when these vector spaces are parametrized over points in a space X, i.e. we have vector bundles. Then the kernel and cokernel are vector bundles over X and their formal difference lies in $K(X)$. It turns out, it is stable under perturbing the map T.

Let H, H' be Hilbert spaces. The linear maps $T : H \to H'$ equipped with the operator norm topology form a complete i.e. Banach space. The OMT guarantees that a bijective linear map has a continous inverse; moreover, the invertible maps form a contractible open subset. (why? something about constructing a ball of invertible operators around any invertible one by using the power series for $1/x + 1$. Idea is that if $T = 1 - A$ is close to the identity, then $1 + T + T^2 + \ldots$ converges and serves as an inverse)

Definition 5.35 (Fredholm operators): *A continous linear map between Hilbert spaces T is Fredholm if its kernel and cokernel are finite dimensional. These operators have a well-defined index*

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T$$

Definition 5.36 (Canonical open cover): *For a finite dimensional $W \subset H'$ we define the space of Fredholm maps transverse to it:*

$$\mathcal{O}_W = \{T \in \operatorname{Fred}(H, H') | T(H) + W = H'\}$$

Transversality to a fixed W is an open condition: it is equivalent to the sequence

$$T^{-1}W^{\perp} \to H \to H' \to H'/W$$

being an isomorphism. Moreover, any Fredholm operator is transverse to a finite dimensional W since it has finite dimensional cokernel. Hence, the $\mathcal{O}_W$ form an open cover of the space of Fredholm operators. In fact, if X is a compact space and we have a continous map $X \to \mathrm{Fred}(H,H')$, then the cover $T^{-1}\mathcal{O}_W$ has a finite subcover indexed by $W_1,...,W_n$ and if we take their sum $W = \oplus W_i$, then $T(X) \subset \mathcal{O}_W$ with W finite dimensional!

On the contractible open sets $\mathcal{O}_W$ we have locally trivial vector bundles

$$K_W \to \mathcal{O}_W$$

whose fiber at T consists of the vector space $T^{-1}W$. This is topologized as a subspace of $\mathrm{Hom}(H,H')\times H$. Can this be glued to a vector bundle on the whole of $\mathrm{Fred}(H,H')$? In any case, this tells us that the index function

$$\mathrm{ind} : \mathrm{Fred}(H,H') \to \mathbb{Z},\ T \mapsto \dim\ker T - \dim\operatorname{coker} T$$

is locally constant, i.e. invariant under perturbations of T, since it is equal to $\dim T^{-1}W - \dim W$.

Definition 5.37 (Generalized index): *If we put $\mathcal{F} = \mathrm{Fred}(H,H)$, then given any X compact, a map $F : X \to \mathcal{F}$ lands in some $\mathcal{O}_W$, so we can consider the vector bundle F^*K_W over X. This allows us to define the index map, which is an isomorphism:*

$$[X,\mathcal{F}] \simeq K(X)$$
$$F \mapsto F^*K_W - \underline{W}$$

This shows that the space of Fredholm operators classifies K-theory.

Why is this well-defined? Well, given W,V such that $F(X) \subset \mathcal{O}_W, \mathcal{O}_V$ then the same holds for their sum and so we only need to check it for inclusions $V \subset W$. But then, we have a SES of vector bundles

$$0 \to F^*K_V \to F^*K_W \to \underline{W}/\underline{V} \to 0$$

This splits and adding $\underline{V}$ we get

$$F^*K_W \oplus \underline{V} \simeq F^*K_V \oplus \underline{W}$$

The invariance under homotopy is given by the fact that $X \times [0,1]$ is also compact, hence a homotopy also has image in a given $\mathcal{O}_W$ and then we can use concordance.

Note: this is also an isomorphism of abelian groups, where we compose maps pointwise (we chose $H' = H$ to make it into a monoid). The index map is a homomorphism, since

$$\mathrm{ind}(T_2 \circ T_1) = \dim T_1^{-1}T_2^{-1}W - \dim W = \dim T_1^{-1}T_2^{-1}W - \dim T_2^{-1}W + \dim T_2^{-1}W - \dim W = \mathrm{ind}\, T_1 + \mathrm{ind}\, T_2$$

Remark: Now, we can ask whether the two classifying spaces are the same, and in fact they are, up to homotopy! The $\mathbb{Z}$ is keeping track of the connected components, so in fact we have that

$$BU \simeq \mathcal{F}_0$$

5.3.3.6 The index of an elliptic operator

Need to define what an elliptic operator is and why it is Fredholm.

Theorem 5.38 (Kuiper's theorem): *All Hilbert space bundles are trivial, since $GL_\infty(H)$ is contractible. Has a proof in Freed's notes.*

Suppose we have a map of bundles with fibers H, i.e. with structure group $GL(H)$. Then they are both trivial, so if our morphism $f : B_1 \to B_2$ is Fredholm on every fiber, it gives us a map $X \to \mathcal{F}$. This only depends on the homotopy classes involved, and hence this gives us an element of $K(X)$.

More generally, consider vector bundles E, F over $M \times X$ and a family of elliptic operators d, i.e. something that restrics to an elliptic operator $d_x : \Gamma(E_x) \to \Gamma(F_x)$ for all $x \in X$. The index of this is an element of $K(X)$. Given Q over $M \times X$, we can form an elliptic family d_Q from $E \otimes Q$ to $F \otimes Q$ whose index index_Q is well defined, which extends linearly to a homomorphism

$$\text{index}_d : K(M \times X) \to K(X)$$

5.3.3.7 Proof of Bott periodicity using McDuff's quasifibration

Basically, there is a fibration

$$\begin{tikzcd} \mathbb{Z} \times BU_\infty \arrow[r] & \mathcal{H} \arrow[d] \\ & U_\infty \end{tikzcd}$$

Check Jack's notes, or McDuff's paper.

5.3.3.8 Proof of Bott periodicity using Fredholm operators

Basically, we wish to construct an inverse to the "multiplication by the Bott element" map. This will be given by the index of an elliptic operator, such that the index of the Bott element is 1.

The idea now is to consider the del bar operator on bundles over $\mathbb{CP}^1$. This is an elliptic operator and has

$$\ker \overline{\partial}_E = H^0(\mathbb{CP}^1, \mathcal{E})$$
$$\text{coker} \overline{\partial}_E = \ker \overline{\partial}_E^* = H^1(\mathbb{CP}^1, \mathcal{E})$$

We need Serre duality to deduce this last calculation. This is given by the integration pairing

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^*) \to \mathbb{C}$$

More precisely, we have a pairing

$$\mathcal{E} \otimes \mathcal{E}^* \otimes K_X \to K_X$$

which, since $H^n(X,K_X) \simeq \mathbb{C}$ by integration, induces an isomorphism

$$H^p(X,\mathcal{E}) \simeq H^{n-p}(X,\mathcal{E}^* \otimes K_X)^*$$

This happens because the adjoint of $\overline{\partial}_E$ is defined as $\overline{\partial}_E^* := (-1)^q *_E^{-1} \circ \overline{\partial}_{K_X \otimes E^*} \circ *_E : A^{o,q}(E) \to A^{0,q-1}(E)$, i.e. we have the sequence

$$\Omega_X^{0,q} \otimes E \xrightarrow{*_E} \Omega_X^{n,n-q} \otimes E^* = \Omega_X^{n,0} \otimes \Omega_X^{0,n-q} \otimes E^* = \Omega_X^{0,n-q} \otimes K_X \otimes E^* \xrightarrow{\overline{\partial}_{K_X \otimes E^*}} \Omega_X^{0,n-q+1} \otimes K_X \otimes E^* \xrightarrow{*_E^{-1}} \Omega_X^{0,q-1} \otimes E$$

where the del bar operates on sections, not the bundles.

By considering the trivial bundle and the tautological bundle $\mathcal{O}(-1) = H$ we get a virtual bundle of dimension 0 which is $1 - H \in \tilde{K}^0(\mathbb{CP}^1)$ and its index is 1. Why? Well,

$$\begin{gathered}
H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}) = \mathbb{C} \\
H^1(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}) = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}^* \otimes K_{\mathbb{CP}^1})^* = H^0(\mathbb{CP}^1, \mathcal{O}(-2))^* = 0 \\
H^0(\mathbb{CP}^1, \mathcal{O}(-1)) = 0 \\
H^1(\mathbb{CP}^1, \mathcal{O}(-1)) = H^0(\mathbb{CP}^1, \mathcal{O}(-1)^* \otimes K_{\mathbb{CP}^1})^* = H^0(\mathbb{CP}^1, \mathcal{O}(-1)) = 0
\end{gathered}$$

Then we finish up by considering:

$$\tilde{K}^0(X) \otimes \tilde{K}^0(\mathbb{CP}^1) \xrightarrow{\boxtimes} \tilde{K}^0(X \wedge \mathbb{CP}^1) \xrightarrow{inc} \tilde{K}^0(X \times \mathbb{CP}^1) \xrightarrow{\text{index}_{\overline{\partial}}} \tilde{K}^0(X)$$

First map is external product, second is inclusion as k theory of product is sum of k theory of wedge and k theories of separate bits, last map is the index homomorphism. This is hopefully the map alpha?

5.3.4 Proof of the fundamental product theorem

Non-examinable. Will add proof later.

We would like to show that there is an isomorphism given by the composition

$$K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K^0(X) \otimes K^0(\mathbb{CP}^1) \to K^0(X \times \mathbb{CP}^1)$$

where the second map is the cup product.

The proof has the following steps:

- Give a generalized clutching construction for bundles over $X \times \mathbb{CP}^1$ given by sections of the endomorphism bundle $\mathrm{End}(X \times S^1)$
- Analyze these clutching functions and their properties. Namely, show that any cluthcing function can be approximated by a Laurent one, then replace Laurent series by polynomials by mutliplying by a high enough power of z and finally reduce polynomials to linear functions. Then, analyse the linear clutching functions by considering an eigenvalue decomposition.
- There is an ambiguity in the reduction to linear functions, which is taken care of by K-theory!

5.3.4.1 The generalized clutching construction

In the classical clutching contruction, one looks at a bundle E over $\mathbb{CP}^1$ and considers that it is trivial over the upper and lower hemisphere, and glued along a family of matrices along the equator, which is given by a map $S^1 \to \mathrm{GL}(n, \mathbb{C})$. We would like to replace $\mathrm{GL}(n, \mathbb{C})$ with automorphisms of E_x, where E is a bundle over an arbitrary space X - in the classical case, we just have the trivial bundle over a single point.

More precisely, given $E \to X$ inducing a bundle $E \times S^1 \to X \times S^1$ we consider $f \in \mathrm{End}(E \times S^1)$ which can be thought of as a section of the endomorphism bundle, in other words as S^1-parametrized families of endomorphisms $f(x,z) : E_x \to E_x$. There is the following picture

$$\begin{tikzcd} E \times S^1 \arrow[rr, "f"] \arrow[dr, "\pi \times 1"'] & & E \times S^1 \arrow[dl, "\pi \times 1"] \\ & X \times S^1 & \end{tikzcd}$$

We denote by $[E, f]$ the resulting bundle starting with the data E and $f \in \mathrm{Aut}(E \times S^1)$. One can show that every bundle over $X \times \mathbb{CP}^1$ arises like this. Furthermore, $[1, z] = H$, where 1 is the trivial bundle and z means multiplication by z. More generally, we have the following properties (where

$\pi_2 : X \times \mathbb{CP}^1 \to \mathbb{CP}^1$):

$$[E_1, f_1] \oplus [E_2, f_2] \simeq [E_1 \oplus E_2, f_1 \oplus f_2]$$
$$[E, z^n f] \simeq [E, f] \otimes \pi_2^* H^n$$

5.3.4.2 Approximation by Laurent series

We call a clutching function a Laurent clutching function if it looks as follows:

$$f(x,z) = \sum_{|k| \le n} a_k(x) z^k$$

Here, $a_k(x) : E_x \to E_x$ are linear endomorphisms and z acts by scalar multiplying the vectors. Note that a_k can be noninvertible, while f is still invertible!

Given f, we can construct a Laurent series with terms given by

$$a_k(x) = \frac{1}{2\pi i} \int_{S^1} z^{-k} f(x,z) \frac{dz}{z} = \frac{1}{2\pi} \int_{S^1} e^{-ik\theta} f(x, e^{i\theta}) d\theta$$

Here, we are integrating over the space of linear maps $E_x \to E_x$. The Cesaro terms of this series then converge to f.

5.3.4.3 Reduction to polynomials and linear functions

The next step is to reduce to polynomials. But given a Laurent polynomial function f, we have that $[E, f] = [E, pz^{-n}] \simeq [E, p] \otimes \pi_2^* H^{-n}$, for a polynomial p. Now we reduce to linear polynomials using the following lemma:

Lemma 5.39 (Reduction to linear functions): *Given a polynomial clutching function p of degree at most n, we have that*

$$[E, p] \oplus [E^{\oplus n}, Id] \simeq [E^{\oplus n+1}, L^n p]$$

for a linear clutching function $L^n p$. If we change $L^n p$ to some higher $L^m p$, the difference in K-theory is the same. In other words, we put unambiguously

$$[E, p] = [E^{\oplus n+1}, L^n p] - [E^{\oplus n}, Id] \in K^0(X \times \mathbb{CP}^1)$$

Proof. We define $L^n p$ via

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -z & Id & 0 & \dots & 0 & 0 \\ 0 & -z & Id & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ 0 & 0 & 0 & \dots & -z & Id \end{pmatrix}$$

□

Here, $p = \sum a_n z^n$.We think of the (i,j) entry as giving the (i,j) component of the map $E^{\oplus n+1} \to E^{\oplus n+1}$. This can be described by the product

$$\begin{pmatrix} Id & a_1^* & a_2^* & \dots & a_{n-1}^* & a_n^* \\ 0 & Id & 0 & \dots & 0 & 0 \\ 0 & -z & Id & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & Id \end{pmatrix} \begin{pmatrix} p & 0 & 0 & \dots & 0 & 0 \\ 0 & Id & 0 & \dots & 0 & 0 \\ 0 & 0 & Id & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 & 0 & \dots & 0 & 0 \\ -z & Id & 0 & \dots & 0 & 0 \\ 0 & -z & Id & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ 0 & 0 & 0 & \dots & -z & Id \end{pmatrix}$$

All of these are invertible, and the outer terms can be homotoped out via $Id + tN$, where N is nilpotent. Then, the middle bit describes $[E^{\oplus n+1}, p \oplus Id] = [E^1, p] \oplus [E^{\oplus n}, Id]$.

5.3.4.4 Analysis of linear clutching maps

We finish up by the following lemma:

Lemma 5.40 (Analysis of linear clutching functions): *Any linear clutching function $p = az + b$ can be homotoped to $b = Id$. Then there is a splitting $E = E_+ \oplus E_-$ such that $[E, p] \simeq [E_+, Id] \oplus [E_-, z]$*

5.4 More on characteristic classes and K-theory

5.4.1 The Chern character

First, we do a quick review of symmetric polynomials. Recall that the ring of invariant polynomials under the action of the symmetric group is

$$\mathbb{Z}[x_1,...,x_n]^{S_n} = \mathbb{Z}[e_1,...,e_n]$$

where the e_i are the elementary symmetric polynomials. Thus, if $p_k = x_1^k + ... + x_n^k$ it is expressible in terms of the e_i and we put

$$p_k = \overline{p}_k(e_1,...,e_n)$$

Lemma 5.41 (Power sum polynomials identity):

$$p_n - e_1 p_{n-1} ... \pm n e_n = 0$$

Proof. Look at $\prod x_i + t = \sum t^i e_i$ and substitute $t = -x_j$ and sum. □

Now, suppose that E is decomposable as a sum of line bundles $\oplus L_i$. Then

$$c(E) = \prod (1 + c_1(L_i)) = \sum e_i(c_1(L_1),...,c_1(L_n))$$

In other words, $c_i(E) = e_i(c_1(L_1),...,c_1(L_n))$. We seek a formula like this which holds for an arbitrary E, which should exist, by the splitting principle.

Definition 5.42 (Chern character): *We have a homomorphism*

$$\mathrm{ch} : K^0(X) \to H^\bullet(X;\mathbb{Q})$$

such that on line bundles it is $\mathrm{ch}(L) = \exp(c_1(L))$ *and more generally it is defined as*

$$\mathrm{ch}_k(E) := \frac{1}{k!}\overline{p}_k(c_1(E),...,c_n(E))$$

This extends linearly to the rational cohomology.

The idea is that

$$\mathrm{ch}(L_1 \oplus ... \oplus L_n) = \sum \mathrm{ch}(L_i) = \sum \exp(c_1(L_i)) = \sum_{i,j} \frac{1}{j!} c_1(L_i)^j = \sum_j \frac{1}{j!} p_j(c_1(L_1),...,c_1(L_n)) = \sum_j \frac{1}{j!}\overline{p}_j(c_1(E),...,c_n(E))$$

We have that $\mathrm{ch}_0(E) = \dim E, \mathrm{ch}_1(E) = c_1(E)$. Can extend this to K^{-1} via

$$K^{-1}(X) \simeq \tilde{K}^{-1}(X) = \tilde{K}^0(\Sigma X) \xrightarrow{\mathrm{ch}} \tilde{H}^{ev}(\Sigma X;\mathbb{Q}) \simeq H^{odd}(X;\mathbb{Q})$$

In fact, we will show that the Chern character is an isomorphism after tensoring with $\mathbb{Q}$, so rationally, K-theory and cohomology are the same!

Here are the first few Chern characters:

$$\mathrm{ch}_0(E) = \dim(E), \mathrm{ch}_1(E) = c_1(E), \mathrm{ch}_2(E) = \frac{1}{2}(c_1(E)^2 - 2c_2(E))$$

Lemma 5.43 (Chern character lands in integral cohomology of spheres): *The Chern characters*

$$\mathrm{ch} : \tilde{K}^0(S^{2n}) \to \tilde{H}^{ev}(S^{2n}; \mathbb{Q})$$
$$\mathrm{ch} : \tilde{K}^0(S^{2n+1}) \to \tilde{H}^{odd}(S^{2n+1}; \mathbb{Q})$$

are isomorphisms onto the integral part of the cohomology ring.

Proof. Firstly, the odd case reduces to the even case. For $n = 1$, we are looking at the Chern character from $\tilde{K}^0(\mathbb{CP}^1) = \mathbb{Z}[H-1]$ to $H^2(S^2; \mathbb{Q})$. We see that

$$\mathrm{ch}(H-1) = \exp(c_1(H)) - 1 = c_1(H) = -x$$

which is the integral generator of the second cohomology, so it checks out. For an arbitrary even sphere, the Fundamental Product theorem tells us that that the box product

$$\tilde{K}^0(S^2) \otimes \tilde{K}^0(S^{2n-2}) \to \tilde{K}^0(S^{2n})$$

is an isomorphism. So by induction,

$$\tilde{K}^0(S^{2n}) = \mathbb{Z}[(H-1)^{\boxtimes n}$$

. Thus, $\mathrm{ch}(\tilde{K}^0(S^{2n}))$ is generated by $(c_1(H))^{\boxtimes n}$ which is exactly the integral part.

□

Remark: For an arbitrary space the Chern character need not be integral!

Theorem 5.44 (Chern character is an isomorphism between rational K-theory and cohomology): *The Chern character* $\mathrm{ch} : K^\bullet \to H^\bullet(X; \mathbb{Q})$ *is a homomorphism of* $\mathbb{Z}/2$ *graded rings and furthermore* $\mathrm{ch} \otimes \mathbb{Q}$ *is an isomorphism when X has the homotopy type of a finite CW complex.*

Proof. To show that it is a homomorphism, the only tricky part is to see whether everything commutes with the Bott isomorphism, i.e. the case $\bullet = i = j = -1$. In other words, we need to check that applying the external product, then the Chern character, is the same as applying the

Chern character and then the cup product. This comes down to the following diagram:

$$
\begin{tikzcd}
\tilde{K}^{-1}(X)\otimes\tilde{K}^{-1}(X)\otimes\tilde{K}^{-1}(X) \arrow[d, "="] & & & \\
\tilde{K}(S^1\wedge X)\otimes\tilde{K}^0(S^1\wedge X) \arrow[r, "\boxtimes"] \arrow[d, "\mathrm{ch}\otimes\mathrm{ch}"'] & \tilde{K}^0(S^1\wedge X\wedge S^1\wedge X) \arrow[r, "(\Sigma^2\Delta)^*"] \arrow[d, "\mathrm{ch}"'] & \tilde{K}^0(S^2\wedge X) \arrow[d, "\mathrm{ch}"'] & \tilde{K}^0(X) \arrow[l, "-((H-1)\boxtimes -)"', "\text{Bott map}"] \arrow[d, "\mathrm{ch}"'] \\
\tilde{H}^{ev}(X;\mathbb{Q})\otimes\tilde{H}^{ev}(X;\mathbb{Q}) \arrow[r, "\boxtimes"] & \tilde{H}^{ev}(S^1\wedge X\wedge S^1\wedge X) \arrow[r, "(\Sigma^2\Delta)^*"] & \tilde{H}^{ev}(S^2\wedge X) & \tilde{H}^{ev}(X) \arrow[l, "x\boxtimes -"'] \\
\tilde{H}^{odd}(X;\mathbb{Q})\otimes\tilde{H}^{odd}(X;\mathbb{Q}) \arrow[u, "\text{suspension iso}", "(t\boxtimes -)\otimes(t\boxtimes -)"'] \arrow[rrru, bend right, "-\cup -"'] & & &
\end{tikzcd}
$$

The commutativity comes down to the fact that we are inserting swap signs and the fact that the generator $t \in \tilde{H}^1(S^1;\mathbb{Q})$ has $t \boxtimes t = -x$.

For the second part, we know it holds for spheres by 5.43, and then we compare long exact sequences and use induction and the five-lemma. □

5.4.2 K-theory and the K-theory of complex projective space

By example 5.3.2.2, we have that for $H = \overline{\gamma}_{\mathbb{C}}^{1,n+1}$, $(H-1)^{n+1} = 0$. This in fact tells us everything about the K-theory of $\mathbb{CP}^n$;

Theorem 5.45 (K-theory of $\mathbb{CP}^n$):

$$K^0(\mathbb{CP}^n) \simeq \mathbb{Z}[H]/(H-1)^{n+1}, K^{-1}(\mathbb{CP}^n) = 0$$

Proof. We go by induction, and use the LES of a pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$:

$$
\begin{tikzcd}
K^0(\mathbb{CP}^{n-1}) \arrow[d, "\partial"'] & K^0(\mathbb{CP}^n) \arrow[l, "\iota^*"'] & \tilde{K}^0(S^{2n}) \simeq \mathbb{Z} \arrow[l, "q^*"'] \\
0 = \tilde{K}^{-1}(S^{2n}) \arrow[r] & K^{-1}(\mathbb{CP}^n) \arrow[r] & K^{-1}(\mathbb{CP}^{n-1}) = 0 \arrow[u, "\partial"']
\end{tikzcd}
$$

This takes care of $K^{-1}(\mathbb{CP}^{n-1}) = 0$. We get a short exact sequence for K^0 and see that $(H-1)^n \in \ker\iota^* = \operatorname{im} q^*$, so $(H-1)^n = q^*(Y), Y \in \tilde{K}^0(S^{2n}) = \mathbb{Z}\{(H-1)^{\boxtimes n}\}$.

Now, q^* is injective and $q^*\,\mathrm{ch}(H-1)^{\boxtimes n} = c_1(H)^n$, by the lemma 5.43. On the other hand, we have that

$$q^*\,\mathrm{ch}(Y) = \mathrm{ch}(q^*Y) = \mathrm{ch}(H-1)^n = (\exp(-x)-1)^n = (-x)^n$$

so Y also generates $\tilde{K}^0(S^{2n})$ by injectivity of q^*, so $(H-1)^n$ generates $\ker\iota^*$ and the result follows. □

5.4.3 The projective bundle formula and Chern classes for K-theory

We now prove an analogue of the projective bundle formula in cohomology, allowing us in the same way to define Chern classes.

Theorem 5.46 (Projective bundle formula for K-theory): *Suppose E is a d-dimensional complex vector bundle over a compact Hausdorff X. Then we have an isomorphism*

$$K^j(X)\otimes\mathbb{Z}\{1,H_E,...,H_E^{d-1}\}\simeq K^j(\mathbb{P}(E))$$

Proof. As in the case of cohomology, once the case of trivial bundles is done, one can just use LES's, induction and the five lemma. For trivial E this reduces to computing $K^j(\mathbb{P}(X\times\mathbb{C}^d)) = K^j(X\times\mathbb{CP}^{d-1})$. However, note that the bundle L_E in this case is precisely equal to π_2^*H, where $\pi_2 : X\times\mathbb{CP}^{d-1}\to\mathbb{CP}^{d-1}$ is the second projection and hence the isomorphism we want is equivalent to the cup product

$$K^j(X)\otimes K^0(\mathbb{CP}^{d-1})\to K^j(X\times\mathbb{CP}^{d-1})$$

When $d=1,2$ this comes down to the Fundamental Product theorem. For arbitrary d we proceed by induction: the LES of the pair $(\mathbb{CP}^{d-1},\mathbb{CP}^{d-2})$ splits, so stays exact upon tensoring with $K^j(X)$. We also consider the LES of $(X\times\mathbb{CP}^{d-1},X\times\mathbb{CP}^{d-2})$. Hence, we get a situation as follows:

$$\begin{tikzcd}
0 & K^j(X)\otimes K^0(\mathbb{CP}^{d-2}) \arrow[l] \arrow[d, "\simeq"'] & K^j(X)\otimes K^0(\mathbb{CP}^{d-1}) \arrow[l] \arrow[d] & K^j(X)\otimes\tilde{K}^0(\mathbb{CP}^{d-1}/\mathbb{CP}^{d-2}) \arrow[l] \arrow[d] & 0 \arrow[l] \\
& K^j(X\times\mathbb{CP}^{d-2}) & K^j(X\times\mathbb{CP}^{d-1}) \arrow[l] & \tilde{K}^j(X\times\mathbb{CP}^{d-1}/X\times\mathbb{CP}^{d-2}) \arrow[l] &
\end{tikzcd}$$

By the five lemma, it is enough to show that the vertical map on the right is an isomorphism which is precisely $d-1$ times the Bott isomorphism:

$$\tilde{K}^j(X_+)\otimes\tilde{K}^0(S^{2(d-1)})\to\tilde{K}^j(X\wedge S^{2(d-1)})$$

□

Remark: There is a generalization of the projective bundle formula, called the Leray-Hirsch theorem, a proof of which can be found in Hatcher's book on K-theory.

As a corollary to the projective bundle formula, we can apply the same procedure as in 5.14 and get:

Theorem 5.47 (Splitting principle for K-theory): *For any complex vector bundle $E\to X$ over a compact Hausdorff X, there is a space $F(E)$ and a map $f : F(E)\to X$ such that f^*E is a sum of line bundles and f^* is injective on K-theory.*

Now we would like to find the classes in K-theory which give a relation between H_E^d and the lower order terms, in exact analogy with what we did for cohomology. To do this, consider

$$\Lambda_t(E)=\sum_0^\infty\Lambda^i(E)t^i\in K^0(X)[[t]]$$

This has $\Lambda_t(E \oplus F) = \Lambda_t(E)\Lambda_t(F)$ and can be extended linearly by taking formal inverses in the power series ring. It then extends to a map

$$\Lambda_t : (K^0(X), \oplus, 0) \to (K^0(X)[[t]]^\times, \times, 1)$$

We then have

Theorem 5.48 (Chern classes for K-theory): *There is a relation*

$$\sum (-1)^i p^* \Lambda^i(\overline{E}) H_E^{d-i} = 0 \in K^0(\mathbb{P}(E))$$

Proof. By the reasoning in 5.14, we have that L_E sits inside p^*E, where $p : \mathbb{P}(E) \to X$ is the projection map. Thus, $p^*E = L_E \oplus W$ for some $d-1$-dimensional complement W. Thus,

$$\Lambda_t(W) = \frac{\Lambda_t(p^*E)}{\Lambda_t(L_E)} = p^*\Lambda_t(E)(1 - L_E + L_E^2 - ...)$$

Comparing the coefficients of t^d and noting that the d-th exterior power of W is zero, for dimension reasons, we see that

$$0 = \Lambda^d W = \sum p^* \Lambda^i(E)(-L_E)^{d-i}$$

Conjugating, we get the relation in the theorem. □

5.4.4 Thom isomorphism, Euler class and Gysin sequence for K-theory

In the classical situation of the Thom isomorphism, we take the complement of the zero section $E^\#$. However, the problem is that $E/E^\#$ is not compact Hausdorff, so we cannot talk about its K-theory. Equipping a complex vector bundle with a Hermitian inner product, we redefine the Thom space to be

$$Th(E) := \mathbf{D}(E)/\mathbf{S}(E)$$

We then get an analogue of the Thom isomorphism theorem in K-theory:

Theorem 5.49 (Thom isomorphism): *There are Thom classes $\lambda_E \in \tilde{K}^0(Th(E))$ such that*

$$K^i(X) \simeq K^i(D(E)) \xrightarrow{\lambda_E} \tilde{K}^i(Th(E))$$

is an isomorphism and these classes are natural under the induced map on the Thom spaces, i.e. if $f : X' \to X$ is a map we have $\hat{f} : E' \to E$ such that

$$Th(\hat{f})^* \lambda_E = \lambda_{E'}$$

Furthermore, we require the normalization property: $\lambda \in \tilde{K}^0(Th(\mathbb{C}^n)) = \tilde{K}^0(S^{2n})$ is the generator $(1-H)^{\boxtimes n}$

Proof. We have an inclusion $\mathbb{P}(E) \to \mathbb{P}(E \oplus \mathbb{C}_X)$. The map $E \to \mathbb{P}(E \oplus \mathbb{C}_X)$ which adds a 1 to the last entry is a homeomorphism onto the complement of the image of $\mathbb{P}(E)$. Thus, we can identify

$\mathbb{P}(E\oplus\mathbb{C}_X)/\mathbb{P}(E)\simeq E_+$. This allows us to write a radial homeomorphism $Th(E)\simeq E_+$, i.e. we can think of points in $Th(E)$ either as in the open unit disk contracting to E, or on the boundary of the disk, which is the point at infinity. We then write the LES of the pair $(\mathbb{P}(E\oplus\mathbb{C}_X),\mathbb{P}(E))$ to get:

$$\begin{tikzcd}
K^0(\mathbb{P}(E)) \arrow[d, "\partial=0"'] & K^0(\mathbb{P}(E\oplus\mathbb{C}_X)) \arrow[l, two heads, "\iota^*"'] & \tilde{K}^0(Th(E)) \arrow[l, hook', "q^*"'] \\
\tilde{K}^{-1}(Th(E)) \arrow[r, "q^*"'] & K^{-1}(\mathbb{P}(E\oplus\mathbb{C}_X)) \arrow[r, "\iota^*"'] & K^{-1}(\mathbb{P}(E)) \arrow[u, "\partial=0"']
\end{tikzcd}$$

By the projective bundle formula for K-theory, the maps ι^* are surjective and hence the boundary maps are zero and q^* is injective. The projective bundle formula also tells us that the K-theory of a projectiviation is generated by the K-theory of the base, together with adjoining a symbol which obeys the relations

$$K^i(\mathbb{P}(E))=K^i(X)[H_E]/(\sum_0^d p^*\Lambda^i(\overline{E})H_E^{d-i})$$
$$K^i(\mathbb{P}(E\oplus\mathbb{C}_X))=K^i(X)[H_{E\oplus\mathbb{C}}]/(\sum_0^{d+1} p^*\Lambda^i(\overline{E\oplus\mathbb{C}_X})H_{E\oplus\mathbb{C}_X}^{d+1-i})$$

By exactness, the K-theory of the Thom space is the kernel of the map ι^*. Let's write H for all of the classes for convenience. Then the relations become as follows, using $H=L^{-1}$:

$$\sum_0^d p^*\Lambda^i(\overline{E})H_E^{d-i}=H^d\Lambda_{-L}(\overline{E})$$
$$\sum_0^{d+1} p^*\Lambda^i(\overline{E\oplus\mathbb{C}_X})H_{E\oplus\mathbb{C}_X}^{d+1-i}=H^{d+1}\Lambda_{-L}(\overline{E\oplus\mathbb{C}_X})=H^{d+1}\Lambda_{-L}(\overline{E})\Lambda_{-L}(\overline{\mathbb{C}_X})==H^{d+1}\Lambda_{-L}(\overline{E})(1-L)$$

From this and the fact that H is a unit we see that $\iota^*\Lambda_{-L}(\overline{E})=0$ and hence must come from the image of q^*!We then define the Thom class as the unique $\lambda_E\in\tilde{K}^0(Th(E))$ such that $q^*\lambda_E=\Lambda_{-L}(\overline{E})$. Under these identifications, the Thom isomorphism is:

$$K^0(X)\simeq K^0(X)[L]/(L-1)\simeq\Lambda_{-L}(\overline{E})/(\Lambda_{-L}(\overline{E})(1-L))\simeq\ker\iota^*\simeq Th(E)$$

□

Definition 5.50 (K-theoretic Euler class): *The zero section $s_0:X\to E$ induces a based map $X_+\to Th(E)$ where the points in X get sent to 0 and the basepoint to infinity. Then we define the Euler class as*

$$e^K(E):=s_0^*\lambda_E\in\tilde{K}^0(X_+)=K^0(X)$$

We can calculate it as follows:

Proposition 5.51 (Formula for Euler class):

$$e^K(E)=\Lambda_{-1}\overline{E}$$

Proof. The zero section factors through q as follows:

$$X \xrightarrow{s_0'} \mathbb{P}(E \oplus \mathbb{C}_X) \xrightarrow{q} Th(E)$$

This is the composition $x \mapsto \langle 0 \rangle \oplus \mathbb{C}_x^* \mapsto 0 \in E_x$. We characterised λ_E via $q^*\lambda_E = \Lambda_{-L}(\overline{E})$ and hence we see that

$$e^K(E) = s_0^*\lambda_E = (s_0')^* q^* \lambda_E = (s_0')^* \Lambda_{-L}(\overline{E})$$

However, the tautological bundle $L_{E\oplus\mathbb{C}} \to \mathbb{P}(\overline{E \oplus \mathbb{C}_X})$ pulls back via s_0' to the trivial bundle, as can be seen from the description $x \mapsto \langle 0 \rangle \oplus \mathbb{C}^*$, showing the desired formula. □

Just as in cohomology, the LES of the pair $(\mathbf{D}(E), \mathbf{S}(E))$ transforms into the Gysin sequence using the homotopy equivalence $\mathbf{D}(E) \simeq X$ together with the Thom isomorphism:

$$\begin{tikzcd}
K^0(\mathbf{S}(E)) \arrow[d, "p_!"'] & K^0(X) \arrow[l, "p^*"'] & K^0(X) \arrow[l, "e^K(E)"'] \\
K^{-1}(X) \arrow[r, "e^K(E)"'] & K^{-1}(X) \arrow[r, "p^*"'] & K^{-1}(\mathbf{S}(E)) \arrow[u, "p_!"']
\end{tikzcd}$$

5.4.5 The K-theory of $\mathbb{RP}^n$

We now apply the methods from the previous section to calculate the K-theory of $\mathbb{RP}^n$. Firstly, we need a geometric lemma:

Lemma 5.52 (Lemma): *There is a homeomorphism $\overline{\psi} : \mathbb{RP}^{2n+1} \simeq \mathbf{S}(\gamma_{\mathbb{C}}^{1,n+1} \otimes \gamma_{\mathbb{C}}^{1,n+1})$ which, under the projection map to $\mathbb{CP}^n$, pulls back $\gamma_{\mathbb{C}}^{1,n+1}$ to the complexification $\gamma_{\mathbb{R}}^{1,2n+2} \otimes \mathbb{C}$*

Proof. Let us define

$$\psi : S^{2n+1} \to \mathbf{S}(\gamma_{\mathbb{C}}^{1,n+1} \otimes \gamma_{\mathbb{C}}^{1,n+1})$$

which sends x to the point which is given by the unit vector $x \otimes x$ in the tensor product $\langle x \rangle \otimes \langle x \rangle$. This descends under quotienting by the antipodal map, since $(-1)^2 = 1$, and gives us a map which we call $\overline{\psi}$, whose domain is now $\mathbb{RP}^{2n+1}$. This is surjective, as $l \otimes l$ is spanned by $z \otimes z$ for any nonzero vector z. On the other hand, if $\psi(x) = \psi(y)$ then $\langle x \rangle_{\mathbb{C}} = \langle y \rangle_{\mathbb{C}} =$ and so $x = \alpha y$ and $\alpha^2 = 1$, so $\alpha = \pm 1$ and hence $\overline{\psi}$ is injective. Hence, this is a bijective continous map between compact Hausdorff spaces, hence a homeomorphism. For the last part, we can see that the map sends a line $\langle x \rangle_{\mathbb{R}}$ to its complexification $\langle x \rangle_{\mathbb{C}}$. □

We now proceed to compute the K-theory of $\mathbb{RP}^{2n+1}$. We just showed that it is homeomorphic to the bundle $E = \mathbf{S}(\gamma_{\mathbb{C}}^{1,n+1} \otimes \gamma_{\mathbb{C}}^{1,n+1})$ whose Euler class is given by

$$e^K(E) = \Lambda_{-1}(\overline{E}) = 1 - H^2$$

We now plug this into the Gysin sequence for E:

$$\begin{tikzcd}
K^0(\mathbb{RP}^{2n+1}) \arrow[d, "p_!"] & K^0(\mathbb{CP}^n) = \mathbb{Z}[H]/(H-1)^{n+1} \arrow[l, "p^*"'] & K^0(\mathbb{CP}^n) = \mathbb{Z}[H]/(H-1)^{n+1} \arrow[l, "1-H^2"'] \\
0 = K^{-1}(\mathbb{CP}^n) \arrow[r, "1-H^2"'] & 0 = K^{-1}(\mathbb{CP}^n) \arrow[r, "p^*"'] & K^{-1}(\mathbb{RP}^{2n+1}) \arrow[u, "p_!"']
\end{tikzcd}$$

We thus see that $K^{-1}(\mathbb{RP}^{2n+1}) = \ker(1-H^2)$. But $p(H)(1-H^2)$ is divisible by $(1-H)^{n+1}$ precisely when p is divisible by $(1-H)^n$, so this kernel corresponds to the ideal $(1-H)^n/(1-H)^{n+1} \simeq \mathbb{Z}[(H-1)^n]$. In the lemma, we showed that p pulls back H to the complexification $\gamma_{\mathbb{R}}^{1,2n+2} \otimes \mathbb{C}$. If we define $v := \gamma_{\mathbb{R}}^{1,2n+2} \otimes \mathbb{C} - 1$, we see that $p^*(H-1) = v$. Thus, $K^0(\mathbb{RP}^{2n+1})$ is the cokernel of p^* and is generated by v, subject to the relation $v^{n+1} = 0$, as well as $0 = p^*(1-H^2) = -v^2 - 2v$. Hence,

$$K^0(\mathbb{RP}^{2n+1}) = \mathbb{Z}[v]/(v^{n+1}, v^2 + 2v) = \mathbb{Z}[v]/(2^n v, v^2 + 2v) \simeq \mathbb{Z}\{1\} \oplus \mathbb{Z}/2^n\{v\}$$

subject to $v^2 = -2v$.

This completes the calculation for odd projective spaces. For even projective spaces, we apply induction whilst simultaneously using the exact cycles for $(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n})$ and $(\mathbb{RP}^{2n}, \mathbb{RP}^{2n-1})$:

$$\begin{tikzcd}
\mathbb{Z}/2^{n-1} v \simeq \tilde{K}^0(\mathbb{RP}^{2n-1}) \arrow[d] & \tilde{K}^0(\mathbb{RP}^{2n}) \arrow[l] & \tilde{K}^0(S^{2n}) \simeq \mathbb{Z} \arrow[l] \\
0 = \tilde{K}^{-1}(S^{2n}) \arrow[r] & \tilde{K}^{-1}(\mathbb{RP}^{2n}) \arrow[r] & \tilde{K}^{-1}(\mathbb{RP}^{2n-1}) \simeq \mathbb{Z} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{K}^0(\mathbb{RP}^{2n}) \arrow[d] & \tilde{K}^0(\mathbb{RP}^{2n+1}) \simeq \mathbb{Z}/2^n v \arrow[l] & \tilde{K}^0(S^{2n+1}) = 0 \arrow[l] \\
\mathbb{Z} \simeq \tilde{K}^{-1}(S^{2n+1}) \arrow[r] & \tilde{K}^{-1}(\mathbb{RP}^{2n+1}) \simeq \mathbb{Z} \arrow[r] & \tilde{K}^{-1}(\mathbb{RP}^{2n}) \arrow[u]
\end{tikzcd}$$

By the first sequence, $\tilde{K}^{-1}(\mathbb{RP}^{2n})$ is torsion free, and by the second one it is cyclic, so it is 0 or $\mathbb{Z}$. If it is $\mathbb{Z}$, then the lower right map in the second diagram must be an isomorphism and hence we get an SES

$$0 \to \mathbb{Z}/2^n v \to \tilde{K}^0(\mathbb{RP}^{2n}) \to \mathbb{Z} \to 0$$

On the other hand, we see that in the first sequence, the boundary map must be zero, so we get another SES:

$$0 \to \mathbb{Z} \to \tilde{K}^0(\mathbb{RP}^{2n}) \to \mathbb{Z}/2^{n-1} v \to 0$$

But these two sequences are incompatible, and hence $\tilde{K}^{-1}(\mathbb{RP}^{2n}) = 0$. From this it follows that $\tilde{K}^0(\mathbb{RP}^{2n}) \simeq \mathbb{Z}/2^n$. All in all, we can package this into:

Proposition 5.53 (K-theory of real projective space): *For odd projective space, we have:*

$$K^0(\mathbb{RP}^{2n+1}) \simeq \mathbb{Z}\{1\} \oplus \mathbb{Z}/2^n v, K^{-1}(\mathbb{RP}^{2n+1}) \simeq \mathbb{Z}$$

For even projective space, we have:

$$K^0(\mathbb{RP}^{2n}) \simeq \mathbb{Z}\{1\} \oplus \mathbb{Z}/2^n v, K^{-1}(\mathbb{RP}^{2n}) \simeq 0$$

where $v^2 = -2v$

5.4.6 Adams operations

Theorem 5.54 (Adams operations): *There are natural ring homomorphisms $\psi^k : K^0(X) \to K^0(X)$ such that $\psi^k(L) = L^k$ for line bundles, $\psi^{kl} = \psi^k \circ \psi^l$ and $\psi^p(x) = x^p$ modulo p.*

Proof. For sums of line bundles $E = \oplus L_i$ we have to put

$$\psi^k(E) = \sum L_i^k = p_k(L_i) = \overline{p}_k(e_1(L_1, \dots, L_n), \dots, e_n(L_1, \dots, L_n))$$

However, recall that

$$\Lambda_t E = \prod (1 + L_i t) = \sum e_i(L_1, \dots, L_n) t^i$$

Hence, when E is a sum of line bundles, we have that $e_i(L_1, \dots, L_n) = \Lambda^i(E)$. This motivates the definition

$$\psi^k(E) = \overline{p}_k(\Lambda^1(E), \dots, \Lambda^k(E))$$

which works, by the splitting principle. To check this defines a homomorphism, one only needs to do it for line bundles. The mod p property follows by looking at binomial expansions. □

Lemma 5.55 (Adams operations on spheres): *ψ^k acts on $\tilde{K}^0(S^{2n}) \simeq \mathbb{Z}$ as multiplication by k^n. On the odd spheres, ψ^k acts on $\tilde{K}^{-1}(S^{2n+1}) \simeq \mathbb{Z}$ as multiplication by k^{n+1}.*

Proof. When $n = 1$, we see that $\psi^k(x) = \psi^k(H-1) = H^k - 1 = (1+x)^k - 1 = kx$, where x is the Bott generator. For arbitrary n, the generator of $\tilde{K}^0(S^{2n})$ is $(H-1)^{\boxtimes n}$ and the Adams operations commute with it. The case of odd spheres follows by definition of $\tilde{K}^{-1}$.

□

Warning: The Bott isomorphism does not commute with the Adams operations: instead, we have

$$\psi^k(\beta(\alpha)) = \psi^k((H-1) \boxtimes \alpha) = \psi^k(H-1) \boxtimes \psi^k(\alpha) = k\psi^k(\alpha)$$

Example (Adams operations on real projective space): By our calculation in the previous section, we have that

$$\tilde{K}^0(\mathbb{RP}^n) = \mathbb{Z}/2^N$$

where $v = [\gamma_{\mathbb{R}}^{1,n+1} \otimes \mathbb{C}] - 1$. We see that

$$\psi^k(v) = (v+1)^k - 1 = \begin{cases} v, k \text{ odd} \\ 0, k \text{ even} \end{cases}$$

5.4.7 Hopf invariant one problem

A CW complex with cells in dimension $0, 2n, 4n$ can be described by an attaching map $f : \partial D^{4n} = S^{4n-1} \to S^{2n}$. We call this space X_f and celullar cohomology tells us that it has cohomology $\mathbb{Z}$ in degrees $0, 2n, 4n$. Let $\iota : S^{2n} \to X_f$ be the inclusion and $c : X_f \to S^{4n}$ be the collapse map and let $a, b \in H^\bullet(X_f; \mathbb{Z})$ be such that $b = c^*(u_{4n})$ and $\iota^* a = u_{2n}$ i.e. they correspond to the generators of the cohomology of the spheres. We must have a relation of the sort

$$a \cup a = h(f) b$$

and we call $h(f) \in \mathbb{Z}$ the cohomological Hopf invariant of f. Attaching homotopic maps produces homotopic CW complexes, so this is independent of the homotopy class of f and hence we get the Hopf invariant map

$$h : \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$$

Example (Division algebras): For example, the classical Hopf map $S^3 \to S^2$ describing the CW structure of $\mathbb{CP}^2$ has Hopf invariant ± 1, due to the structure of the cohomology. In other words, the cohomology ring of $\mathbb{CP}^2$ is telling us that $\pi_3(S^2)$ is nontrivial! In fact, can do the same thing with octonions and quaternions. These maps have Hopf invariant 1 and a natural question is whether there are any others, and it turns out that the anwser is no!This is related to the fact that there are only three division algebras over the real numbers.

To simplify things a bit, we are going to change gears and look at the same problem, but replacing Betti cohomology with K-theory. The exact cycle of (X_f, S^{2n}) reduces to a short exact sequence of the form

$$0 \leftarrow \tilde{K}^0(S^{2n}) \xleftarrow{\iota^*} \tilde{K}^0(X_f) \xleftarrow{c^*} \tilde{K}^0(S^{4n}) \leftarrow 0$$

But $\tilde{K}^0$ of even spheres is generated by multiple products of $H-1$ with itself. Let us put $B = c^*(H-1)^{\boxtimes 2n}$. Let's also pick an A such that $\iota^* A = (H-1)^{\boxtimes n}$. We thus have that $A^2 \in \ker \iota^*$ and so similarly as before we can define

$$A^2 = h^K(f) B$$

However, we need to be careful as to whether this is well-defined, as we made a choice in picking A!

Lemma 5.56 (Hopf invariants agree): *$h^K(f)$ does not depend on A. Moreover, it agrees with the cohomological Hopf invariant.*

Proof. We compare both relations using the Chern character. Firstly, $\mathrm{ch}(B) = q^* \mathrm{ch}(H-1)^{2n} = c^* u_{2n} = b$, since the Chern character gives an isomorphism between the K-theory and integer cohomology of spheres. On the other hand, $\iota^* \mathrm{ch}(A) = \mathrm{ch}(H-1)^n = \iota^* a$. Hence, $\mathrm{ch}(A) = a + qb$

for some $q \in \mathbb{Q}$. Then, $\mathrm{ch}(A^2) = (a+qb)^2 = a^2 = h(f)b = \mathrm{ch}(h(f)(B))$. But the Chern character is injective, so $h^K(f) = h(f)$. $\square$

Theorem 5.57 (Hopf invariant one): *If $f \in \pi_{4n-1}(S^{2n})$ has odd Hopf invariant, then $2n = 2, 4, 8$.*

Proof. We are going to use the Adams operations. Firstly,

$$\psi^k(B) = q^*\psi^k(u_{2n}) = k^{2n}B$$

by 5.55. Similarly,

$$\psi^k(A) = k^n A + \sigma(k)B$$

Now, on one hand

$$(\psi^2 \circ \psi^3)(A) = \psi^2(3^n A + \sigma(3)B) = 6^n A + 3^n\sigma(2)B + \sigma(3)2^{2n}B$$

On the other hand,

$$(\psi^3 \circ \psi^2)(A) = \psi^3(2^n A + \sigma(2)B) = 6^n A + 2^n\sigma(3)B + \sigma(2)3^{2n}B$$

But these should be the same, by 5.54, hence comparing the coefficients of B we get

$$3^n\sigma(2) + \sigma(3)2^{2n} = 2^n\sigma(3) + \sigma(2)3^{2n} \implies \sigma(2)3^n(3^n - 1) = \sigma(3)2^n(2^n - 1)$$

On the other hand

$$h(f)B = A^2 \equiv \psi^2(A) \equiv \sigma(2)B\,(mod\,2)$$

from 5.54 again. Hence, if $h(f)$ is odd, so is $\sigma(2)$ and then all that is left is some elementary number theory, i.e. we must have that $2^n | 3^n - 1$ which is possible only for $n = 1, 2, 4$. $\square$

The Hopf invariant one problem, as mentioned before, is related to the question of finding all division algebra structures over $\mathbb{R}$. If $\mathbb{R}^n$ is a division algebra, then S^{n-1} becomes an H-space with multiplication $(x, y) \mapsto xy/xy$.

When the dimension of the division algebra is odd,we can see the H-multiplication map would induce $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ with $\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$ by comparing with the two separate inclusion maps $S^{2n} \to S^{2n} \times S^{2n}$ and then $\mu^*(\gamma^2) \neq 0$, a contradiction.

On the other hand, when the dimension is even, there is the following trick:

Lemma 5.58 (H-multiplication induces Hopf invariant one): *If g is the H-multiplication on S^{2n-1} then there exists a map $\hat{g} : S^{4n-1} \to S^{2n}$ with Hopf invariant ± 1.*

Proof. The construction of the map is as follows: given g, we decompose $S^{4n-1} = \partial(D^{2n} \times D^{2n}) = \partial(D^{2n}) \times D^2 n \cup D^{2n} \times \partial(D^{2n})$. Moreover, we think of $S^2 n$ as two discs D^{2n}_+, D^{2n}_- with identified boundaries. We then set $\hat{g}(x,y) = |y|g(x, y/|y|) \in D^{2n}_+$ on $\partial(D^{2n}) \times D^2 n$ and $\hat{g}(x,y) = |x|g(x/|x|, y) \in D^{2n}_-$ on $\partial(D^{2n}) \times D^2 n$. This extends g on $\partial(D^{2n}) \times \partial(D^{2n})$ and is continous. We put $f = \hat{g}$ and $\Phi : (D^{4n}, \partial D^{4n}) \to (X_f, S^{2n})$ to be the characteristic map. If e is the H-unit, then we have a big commutative diagram

$$\begin{tikzcd}
\tilde{K}^0(X_f) \otimes \tilde{K}^0(X_f) \arrow[rr, "\otimes"] & & \tilde{K}^0(X_f) \\
\tilde{K}^0(X_f, D^{2n}_-) \otimes \tilde{K}^0(X_f, D^{2n}_+) \arrow[u, "\simeq"] \arrow[rr, "\otimes"] \arrow[d, "\Phi^*\otimes\Phi^*"'] & & \tilde{K}^0(X_f, S^{2n}) \arrow[u] \arrow[d, "\simeq"', "\Phi^*"] \\
\tilde{K}^0(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}) \otimes \tilde{K}^0(D^{2n} \times D^{2n}, D^{2n} \times \partial D^{2n}) \arrow[rr, "\otimes"] \arrow[d, "\simeq"'] & & \tilde{K}^0(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \\
\tilde{K}^0(D^{2n} \times \{e\}, \partial D^{2n} \times \{e\}) \otimes \tilde{K}^0(\{e\} \times D^{2n}, \{e\} \times \partial D^{2n}) \arrow[urr, "\boxtimes"'] & &
\end{tikzcd}$$

The diagonal map is the external product which is a Bott isomorphism. By chasing around, we see that $B \otimes B$ is sent to the image of a generator, and hence the Hopf invariant is ± 1. $\square$

There is a different approach, which we did in the example sheets, namely that a division algebra structure allows one to trivialize the tangent bundle of projective space. One then complexifies and shows that the corresponding K-theory class is $nv + n - 1$. The only way this can vanish is for $2^{\lfloor n/2 \rfloor}$ to divide n which is possible only for $n = 1, 2, 4, 8$.

5.4.8 Todd classes and cannibalistic classes

Given a complex vector bundle $E \to X$, we have a K-theoretic Thom class $\lambda_E \in \tilde{K}^0(Th(E))$, as well as a cohomological Thom class $u_E \in H^{2d}(Th(E); R)$, since E is oriented, being a complex vector bundle.

Definition 5.59 (Correction classes): *We define the Todd class $Td(E)$ to be the cohomology class such that*

$$\mathrm{ch}(\lambda_E) = Td(E) u_E$$

Furthermore, define the k-th cannibalistic class to be the unique class $\rho^k(E) \in K^0(X)$ such that

$$\psi^k(\lambda_E) = \rho^k(E)\lambda_E$$

These classes measure how the Chern character, resp. Adams operations, fail to commute with the Thom isomorphism.

Recall that the Thom isomorphism is given by $x \mapsto p^* x \cup u_E$. We can think of the projection map p as giving a $H^\bullet(X)$-module structure and thus

$$e(E) \cdot u_E = u_E \cup u_E \in \tilde{H}^\bullet(Th(E))$$

The same idea shows that

$$e^K(E) \cdot \lambda_E = \lambda_E \lambda_E$$

But $e^K(E) = \Lambda_{-1}(\overline{E})$. If we take Chern characters on both sides, we get the equation

$$Td(E)^2 e(E) u_E = \mathrm{ch}(\Lambda_{-1}(\overline{E})) Td(E) u_E$$

But the Todd class is invertible, as can be shown by considering the trivial bundle $\mathbb{C}^n \to *$ whose Todd class is just $(H-1)^{\boxtimes n}$, so Td_0 is nonzero and hence by naturality all Todd classes are invertible. Cancelling the terms, we get

$$Td(E)e(E) = \mathrm{ch}(\Lambda_{-1}(\overline{E})) \in H^\bullet(X;\mathbb{Q})$$

When we take E to be a line bundle, we get a formula $Td(L) = \frac{1-\exp(-c_1 L)}{c_1 L}$ and we would like to extend this for all bundles. Put $Q(t) = \frac{1-\exp(-t)}{t} \in \mathbb{Q}[[t]]$.

Lemma 5.60 (Lemma): *The Todd class satisfies $Td(E \oplus E') = Td(E)Td(E')$ and $Td(L) = Q(c_1(L))$ for line bundles. By the splitting principle, this determines it completely.*

Proof. For the tautological bundle γ over $\mathbb{CP}^n$ we have that $\Lambda_{-1}\overline{\gamma} = 1 - \overline{\gamma}$ and hence the formula gives $Td(\gamma)x = 1 - \exp(-x) \in H^\bullet(\mathbb{CP}^n;\mathbb{Q}) = \mathbb{Q}[x]/(x^{n+1})$. Since this is natural under inclusions $\mathbb{CP}^n \subset \mathbb{CP}^{n+1}$ we see that $Q(t)$ is the correct power series. For the other part, we can suppose by the splitting principle that E, E' are sums of line bundles and moreover, by universality, that $X = (\mathbb{CP}^N)^m$ and E, E' are external direct sums $L_1 \boxplus \ldots \boxplus L_n, L_{n+1} \boxplus \ldots \boxplus L_m$ over the respective factors. If we put $x_i = c_1(L_i)$ we have that $H^\bullet(X;\mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_{n+m}]/(x_i^{N+1})$. Thus, applying the same reasoning we get

$$Td(E \oplus E')x_1 \ldots x_{n+m} = \mathrm{ch}(\prod 1 - \overline{L}_i) = \prod (1 - \exp(-x_i))$$

But this is also equal to $Td(E)x_1 \ldots x_n Td(E')x_{n+1} \ldots x_m$. Since this is true for all n, m, N, we get the desired equality. □

Hence,

$$Td(L_1 \oplus \ldots \oplus L_n) = \prod Q(c_1(L_i))$$

The k-th degree part is a symmetric polynomial in the Chern classes, hence can be written as

$$\tau_k(e_1(c_1(L_1), \ldots, c_1(L_n)), \ldots, e_k(c_1(L_1), \ldots, c_1(L_n))$$

By the splitting principle, this is true for all bundles:

$$Td_k(E) = \tau_k(c_1(E), \ldots, c_k(E))$$

The first few τ_k are

$$\tau_0 = 1, \tau_1 = \frac{-x_1}{2}, \tau_2 = \frac{2e_1^2 - e_2}{12}, \tau_3 = \frac{x_1 x_2 - x_1^3}{24}$$

We use Todd classes to prove that there is no correction needed when we take Thom classes of sums:

Lemma 5.61 (Thom classes of sums): *Let $E \to X, E' \to Y$ be two complex vector bundles and $E \boxplus E' \to X \times Y$ their external direct sum. We saw that $Th(E) \simeq E_+$ by radially projecting $[0,1) \simeq [0,\infty]$ and sending the sphere bundle to infinity. With this identification,*

$$Th(E \boxplus E') = (E \boxplus E')_+ \simeq E_+ \wedge E'_+ \simeq Th(E) \wedge Th(E')$$

Then, $\lambda_E \boxtimes \lambda_{E'} = \lambda_{E \boxplus E'} \in \tilde{K}^0(Th(E \boxplus E'))$

Proof. Enough to show this over many copies of $\mathbb{CP}^N$ with E, E' being external sums of tautological bundles.

Firstly, complex projective space has a CW structure with cells of only even dimension (can do this via Morse theory). Moreover, the normal bundle of $\mathbb{CP}^N$ inside $\mathbb{CP}^{N+1}$ is $\overline{\gamma}$, with $\mathbb{CP}^{N+1} - \mathbb{CP}^N$ contractible, so we have a tubular neighbourhood γ and a contractible complement inducing $\mathbb{CP}^{N+1} \simeq Th(\overline{\gamma}) = Th(\gamma)$.

Thus, the Thom spaces of E, E' have only even cells as

$$Th(E \boxplus E') = Th(L_1) \wedge ... Th(L_{n+m})$$

and the Thom spaces of the tautological line bundles are $\mathbb{CP}^N$'s.

But for CW complexes with only even cells, the K-theory is torsion-free so Chern character is injective! So we can just check it on the level of cohomology after applying ch which just gives

$$\mathrm{ch}(\lambda_{L_1} \boxtimes ... \lambda_{L_n}) = Td(L_1) u_{L_1} ... Td(L_n) u_{L_n}$$

But the product of Thom classes is a Thom class, and moreover we showed that the Todd classes are multiplicative, hence the result follows. □

We now move on to cannibalistic classes. By multiplicativity of Thom classes, we immediately get that $\rho^k(E \oplus E') = \rho^k(E)\rho^k(E')$.

Lemma 5.62 (Cannibalistic classes of line bundles): *If E is a complex line bundle, then*

$$\rho^k(E) = 1 + \overline{E} + ... + \overline{E}^{k-1}$$

Proof. We defined λ_E by the relation

$$q^* \lambda_E = \Lambda_{-L}(\overline{E}) \in K^0(\mathbb{P}(E \oplus \mathbb{C})) = K^0(X)[L]/((1-L)\Lambda_{-L}(\overline{E}))$$

where q is the quotient map to $\mathbb{P}(E \oplus \mathbb{C})/\mathbb{P}(E) \simeq Th(E)$. Thus

$$q^* \rho^k(E)(1 - L\overline{E}) = q^* \psi^k \lambda_E = \psi^k(\Lambda_{-L}(\overline{E})) = \psi^k(1 - L\overline{E}) = 1 - (L\overline{E})^k = (1 + L\overline{E} + ... + (L\overline{E})^k)(1 - L\overline{E})$$

But by the defining relation $(1-L)\Lambda_{-L}(\overline{E})=0$, we can eliminate the L's in the sum and get what we want.

$\square$

For example,

$$\rho^2(L_1\oplus\ldots\oplus L_n)=\prod(1+\overline{L}_i)=\sum\Lambda^i(L_1\oplus\ldots\oplus L_n)$$

i.e. $\rho^2(E)=\Lambda_1(\overline{E})$, which looks like the Euler class, but with 1 instead of -1.

5.4.9 Gysin maps and Grothendieck-Riemann-Roch

Usually, given a map $f:M\to N$, we get an induced pullback map on K-theory going from $K(N)$ to $K(M)$. However, sometimes there is a wrong-way map going the opposite direction.

We define a complex orientation on f to be an embedding $\hat{f}:M\xrightarrow{f\times e}N\times\mathbb{R}^k$, together with a complex structure on the normal bundle $\mathcal{N}_{\hat{f}}$.

Given a tubular neighbourhood $\hat{f}(M)\subset U\subset N\times\mathbb{R}^k$ we can collapse the complement of U to get

$$N_+\wedge S^k=(N\times\mathbb{R}^k)_+\xrightarrow{c}U_+\simeq\mathcal{N}_{\hat{f}_+}\simeq Th(\mathcal{N}_{\hat{f}})$$

Definition 5.63 (Gysin maps): *We define the K-theory Gysin map as the composition*

$$\begin{tikzcd}
K^i(M) \arrow[r, "f_!^K"] \arrow[d, "\simeq"'] & K^{i-k}(N) \\
\tilde{K}^i(Th(\mathcal{N}_{\hat{f}})) \arrow[r, "c^*"'] & \tilde{K}^i(N_+\wedge S^k) \arrow[u, "\simeq"']
\end{tikzcd}$$

The vertical arrows are the Thom and suspension isomorphisms, respectively. Note that $k=\dim(M)-\dim(N)$ *modulo 2, since we have a complex structure on the normal bundle.*
On the other hand, we can do the same procedure for cohomology:

$$\begin{tikzcd}
H^i(M) \arrow[r, "f_!^H"] \arrow[d, "\simeq"'] & H^{i+\dim(N)-\dim(M)}(N) \\
\tilde{H}^{i+k+\dim(N)-\dim(M)}(Th(\mathcal{N}_{\hat{f}})) \arrow[r, "c^*"'] & \tilde{H}^{i+k+\dim(N)-\dim(M)}(N_+\wedge S^k) \arrow[u, "\simeq"']
\end{tikzcd}$$

Remark (Relationship with Gysin sequence): When $E\to X$ is a complex vector bundle with stable inverse F, then $\mathbf{S}(E)\to E\oplus F\oplus\mathbb{R}\simeq\mathbb{C}^k\oplus\mathbb{R}$ is a complex orientation for $p:\mathbf{S}(E)\to X$ and $p_!$ is exactly the map appearing in the Gysin sequence.

Remark (Relationship with Poincare duality): In the context of Betti cohomology, when both manifolds are oriented, $f_!$ is just the composition of f_* and two Poincare duality isomorphisms.

We can try to relate the two Gysin maps using the Chern character. The correction lies in the Todd class of the normal bundle:

Theorem 5.64 (Grothendieck-Riemann-Roch):

$$\mathrm{ch}(f_!^K(x)) = f_!^H(\mathrm{ch}(x)Td(\mathcal{N}_{\hat{f}})$$

Proof. Chase through isomorphisms. Both c^* and the suspension isomorphism commute with the Chern character, except for the Thom isomorphism, where we defined the correction class to be the Todd class. □

Example (Complex manifolds): When M, N are both complex manifolds, can put $TM \oplus \mathcal{N} \simeq \mathbb{C}^{k/2} \oplus f^*TN$ as complex bundles from which we get $Td(\mathcal{N}) = \frac{f^*Td(TN)}{Td(TM)}$ and the formula becomes

$$\mathrm{ch}(f_!^K(x)) = f_!^H(\mathrm{ch}(x)\frac{f^*Td(TN)}{Td(TM)})$$

For example, when N is a point an V is a bundle over M , then the Poincare duality argument shows that $f_!^H : H^{\dim M}(M;\mathbb{Q}) \to H^0(*;\mathbb{Q}) \simeq \mathbb{Q}$ is capping with the fundamental class, and so the formula tells us that

$$\langle \mathrm{ch}(V)Td(M)^{-1}, [M]\rangle = \mathrm{ch}_0 f_!^K(V) \in \mathbb{Z}$$

So, nontrivially, this is an integer!

As a further example, let M be a complex 2-fold, so has only two Chern classes $c_1(TM)$ and $c_2(TM) = e(TM)$. We have that

$$Td(TM) = 1 - \frac{c_1}{2} + \frac{2c_1^2 - c_2}{12} \implies Td(TM)^{-1} = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12}$$

Putting V the trivial 1-dimensional bundle, we then must have that

$$\langle c_1^2 + c_2, [M]\rangle \in 12\mathbb{Z}$$

But by Gauss-Bonnet, $\langle c_2, [M]\rangle = \chi(M)$ and so

$$\int_M c_1^2(TM) \equiv -\chi(M) \mod 12$$

5.4.10 The e-invariant

Firstly, note that we have a homomorphism given by suspending

$$\pi_p(Y, y_0) \to \pi_{p+1}(\Sigma Y, y_0)$$

This allows us to construct the stable homotopy groups of a space Y:

$$\pi_p^s(Y) = \varinjlim \pi_{p+k}(\Sigma^k Y, y_0)$$

In this section, we are going to examine the stable homotopy groups of spheres by studying a phenomenon similar to that of the Hopf invariant. Namely, whenever we have a map $f : S^{2n+2k-1} \to S^{2n}$ we can construct a CW complex X_f with cellular cohomology Z in degrees $0, 2n, 2n+2k$, the latter two of which are generated by a, b such that $\iota^* a = u_{2n}, b = c^* u_{2n+2k}$.

As before, in K-theory we get an exact sequence of the form

$$0 \leftarrow \tilde{K}^0(S^{2n}) \xleftarrow{\iota^*} \tilde{K}^0(X_f) \xleftarrow{c^*} \tilde{K}^0(S^{2n+2k}) \leftarrow 0$$

We put $B = c^*(H-1)^{\boxtimes n+k}$ and A some preimage of the generator $(H-1)^{\boxtimes n}$ under ι^*. Upon applying the Chern character, we must have $\mathrm{ch}(B) = b, \mathrm{ch}(A) = a + \lambda b, \lambda \in \mathbb{Q}$. A priori, λ is not well-defined, but if we choose another $A' = A + rB, r \in \mathbb{Z}$ then $\mathrm{ch}(A') = a + \lambda b + rb$, and so λ is well-defined in $\mathbb{Q}/\mathbb{Z}$.

In other words, the case $n = k$ is special, as then there is a relation $A^2 = h^K(f)B$. In this case, this is not allowed, and we get the so-called e-invariant $e(f) \equiv \lambda \mod \mathbb{Z}$.

Lemma 5.65 (e-invariant): *The map*

$$e : \pi_{2n+2k-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}$$

is a homomorphism. Moreover, $e(\Sigma^2 f) = e(f)$, and hence induces a map $\pi_{2k-1}^s \to \mathbb{Q}/\mathbb{Z}$

Proof. Recall that the product fg in the homotopy groups was given by composing with the equatorial collapse map: $S^p \to S^p \vee S^p \xrightarrow{f \vee g} X$ This can also be thought of as the middle line in a square, splitting a big square into smaller rectangles such that all the boundaries map to a single point.

Hence, if we collapse the equator by a map e in the CW complex X_{fg} given by the attaching map fg, we get $X_{f,g} = (S^{2n} \cup_f D^{2n+2k}) \cup_g D^{2n+2k}$, which contains both X_f and X_g. Let's denote this map by ϕ. Now, $X_{f,g}$ is a CW complex with one $2n$ cell and two $2n+2k$ cells, so we get a sequence

$$0 \leftarrow \tilde{K}^0(S^{2n}) \xleftarrow{\iota^*} \tilde{K}^0(X_{f,g}) \xleftarrow{c^*} \tilde{K}^0(S^{2n+2k} \vee S^{2n+2k}) \leftarrow 0$$

The right hand group is just $\tilde{K}^0(S^{2n+2k}) \oplus \tilde{K}^0(S^{2n+2k}) \simeq \mathbb{Z}^2$, so we can denote by B_f, B_g the images under c^* of the generators. We can also pick $A_{f,g}$ such that $\iota^* A_{f,g}$ is a generator. Since the inclusions of X_f, X_g into X_{fg} are natural, i.e. we have a big commutative diagram

$$\begin{tikzcd}
 & & \tilde{K}^0(X_g) \arrow[dl, "\iota^*"'] & \tilde{K}^0(S^{2n+2k}) \arrow[l, "c^*"'] \arrow[d, "p_g^*"] & \\
0 & \tilde{K}^0(S^{2n}) \arrow[l] & \tilde{K}^0(X_{f,g}) \arrow[l, "\iota^*"] \arrow[u, "\iota^*"] \arrow[d, "\iota^*"] & \tilde{K}^0(S^{2n+2k} \vee S^{2n+2k}) \arrow[l, "c^*"] & 0 \arrow[l] \arrow[ul] \arrow[dl] \\
 & & \tilde{K}^0(X_f) \arrow[ul, "\iota^*"] & \tilde{K}^0(S^{2n+2k}) \arrow[l, "c^*"] \arrow[u, "p_f^*"'] &
\end{tikzcd}$$

where all the ι are inclusions, c are collapse maps and p_f, p_g are the projections from a wedge to one summand. In particular, if $a_{f,g}, b_f, b_g$ are the analogous classes in cohomology, then $\mathrm{ch}(A_{f,g}) - a_{f,g}$ is in the kernel of c^* and hence is of the form $\lambda_f b_f + \lambda_g b_g$. However, under the map ϕ, both B_f and B_g pull back to the image of the generator B_{fg}, i.e. we have the commutative diagram

$$\begin{tikzcd}
 & & \tilde{K}^0(X_{fg}) \arrow[dl, "\iota^*"'] & \tilde{K}^0(S^{2n+2k}) \arrow[l, "c^*"'] & \\
0 & \tilde{K}^0(S^{2n}) \arrow[l] & \tilde{K}^0(X_{f,g}) \arrow[l, "\iota^*"] \arrow[u, "\phi^*"] & \tilde{K}^0(S^{2n+2k} \vee S^{2n+2k}) \arrow[l, "c^*"] \arrow[u, "e^*"] & 0 \arrow[l] \arrow[ul]
\end{tikzcd}$$

which again just comes from a commuting diagram in the underlying space maps. In other words, $B_f = c^* p_f^*(\beta)$ where β generates $\tilde{K}^0(S^{2n+2k})$ and hence $\phi^* B_f = \phi^* c^* p_f^*(\beta) = c^* e^* p_f^*(\beta) = c^*(p_f \circ e)^*()\beta$, but $p_f \circ e$ is a homeomorphism, so sends generator to generator; similarly for B_g. If we call $\phi^*(A_{f,g}) = A_{fg}$ which is a choice of preimage, then upon applying the Chern character we get that

$$\mathrm{ch}(A_{fg}) = a + (\lambda_f + \lambda_g)b$$

So for the choice of preimage A_{fg} we get that $\lambda_{fg} = \lambda_f + \lambda_g$. When we go mod $\mathbb{Z}$, there is no longer any ambiguity about the choice and this proves the lemma.

For the last part, one only need notice that the Chern character commutes with the Bott isomorphism. □

We can describe an f explicitly, or implicitly by first producing some CW complex with a cell in dimensions $0, 2n, 2n+2k$. A nice set of examples comes in the following form:

Lemma 5.66 (Lemma): *If $E \to S^{2k}$ is an n-dimensional complex vector bundle, then $Th(E)$ has the homotopy type of a CW complex with one cell in dimensions $0, 2n, 2n+2k$.*

Proof. Let $p : D^{2k} \to S^{2k}$ be the quotient map that collapses the boundary to the south pole S. Then the bundle $p^*\mathbf{D}(E) \simeq D^{2n} \times D^{2k}$ is trivial. Now, we can describe $\mathbf{D}(E)$ using the difference in behaviour between $\mathbf{D}(E_S)$ and $\mathbf{S}(E)$, which is measured by what happens on the boundary of the trivial bundle $p^*\mathbf{D}(E)$. In other words, we have an attaching map

$$\partial(p^*\mathbf{D}(E)) \simeq \partial D^{2k} \times D^2 n \cup D^{2k} \times \partial D^{2n} \simeq p^*\mathbf{D}(E)|_{\partial D^{2k}} \cup p^*\mathbf{S}(E) \xrightarrow{p} \mathbf{D}(E)_S \cup \mathbf{S}(E)$$

Hence, $\mathbf{D}(E)$ is given as the union of $\mathbf{D}(E)_S \cup \mathbf{S}(E)$ along with a $2n+2k$-cell attached along the map and hence the Thom space can be obtained from $\mathbf{D}(E_S)/\mathbf{S}(E_S) \simeq S^{2n}$ by attaching a $2n+2k$–cell. □

Let's put this to use. Recall that $\tilde{K}^0(S^{2k}) = \mathbb{Z}\{(H-1)^{\boxtimes k}\}$, and since reduced K-theory classes have the form $E - \mathbb{C}^n$ for some n-dimensional complex vector bundle E, the same must be true for the generator. In particular, $\mathrm{ch}(E) = n + u_{2k}$. If we look in the sequence

$$0 \leftarrow \tilde{K}^0(S^{2n}) \xleftarrow{\iota^*} \tilde{K}^0(X_f = Th(E)) \xleftarrow{c^*} \tilde{K}^0(S^{2n+2k}) \leftarrow 0$$

then putting $X_f = Th(E)$ we have a canonical choice of A, namely the Thom class λ_E which restricts on each fiber of the Thom space to a generator. The e-invariant is then the coefficient of b in

$$\mathrm{ch}(A) = \mathrm{ch}(\lambda_E) = Td(E)u_E = a + \langle Td_k(E), [S^2k]\rangle b$$

Hence, we are in a situation where we want to find $Td(E)$ but we only know $\mathrm{ch}(E)$. To do this, we need the following lemma

Lemma 5.67 (Todd classes in terms of Chern character): *efine*

$$\log(\frac{1-\exp(-t)}{t}) = \sum_{j\geq 1} d_j \frac{t^j}{j!}$$

Then for any complex vector bundle we have

$$\log(Td(E)) = \sum_{j\geq 1} d_j \,\mathrm{ch}_j(E)$$

Proof. By the splitting principle, only need to verify it for sums of line bundles, where it follows almost by definition:

$$\log(Td(L_1 \oplus \ldots \oplus L_n)) = \log(\prod \frac{1-\exp(c_1(L_i))}{c_1(L_i)}) = \sum_i \sum_j d_j \frac{c_1(L_i)^j}{j!} = \sum_j d_j \,\mathrm{ch}_j(E)$$

□

In our example, $\mathrm{ch}(E) = n + u_{2k}$. Hence,

$$\log(Td(E)) = d_k u_{2k} \implies Td(E) = 1 + d_k u_{2k}$$

as $u_{2k}^2 = 0$.Finally, $u_{2k} \cup u_E = b$ and hence the coefficient of b in $\mathrm{ch}(A)$ must be precisely d_k!

All in all, the e-invariant of the map associated to $Th(E)$ for a complex vector bundle $E \to S^{2k}$ is given by $e(f) \equiv d_k \mod \mathbb{Z}$.